CHAPTER 5

Probability Distributions

5.1. PROBABILITY

Here we define and explain certain terms which are used frequently.

(*a*) **Trial and event.** Let an experiment be repeated under essentially the same conditions and let it result in any one of the several possible outcomes. Then, the experiment is called a *trial* and the possible outcomes are known as *events* or *cases*.

For example: (*i*) Tossing of a coin is a trial and the turning up of head or tail is an event.

(*ii*) Throwing a die is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

(*b*) **Exhaustive events.** The total number of all possible outcomes in any trial is known as *exhaustive events* or *exhaustive cases*.

For example: (*i*) In tossing a coin, there are two exhaustive cases, head and tail.

(*ii*) In throwing a die, there are 6 exhaustive cases, for any one of the six faces may turn up.

(*iii*) In throwing two dice, the exhaustive cases are $6 \times 6 = 6^2$ for any of the 6 numbers from 1 to 6 on one die can be associated with any of the 6 numbers on the other die.

In general, in throwing n dice, the exhaustive cases are 6^n.

(*c*) **Favourable events or cases.** The cases which entail the happening of an event are said to be *favourable* to the event. It is the total number of possible outcomes in which the specified event happens.

For example: (*i*) In throwing a die, the number of cases favourable to the appearance of a multiple of 3 are two *viz.* 3 and 6 while the number of cases favourable to the appearance of an even number are three, *viz.*, 2, 4 and 6.

(*ii*) In a throw of two dice, the number of cases favourable to getting a sum 6 is 5, *viz.*, (1, 5) ; (5, 1); (2, 4); (4, 2); (3, 3).

(*d*) **Mutually exclusive events.** Events are said to be *mutually exclusive* or *incompatible* if the happening of any one of them precludes (*i.e.*, rules out) the happening of all others, *i.e.*, if no two or more than two of them can happen simultaneously in the same trial.

For example: (*i*) In tossing a coin, the events head and tail are mutually exclusive, since if the outcome is head, the possibility of getting tail in the same trial is ruled out.

(*ii*) In throwing a die, all the six faces numbered, 1, 2, 3, 4, 5, 6 are mutually exclusive since any outcome rules out the possibility of getting any other.

(*e*) **Equally likely events.** Events are said to be *equally likely* if there is no reason to expect any one in preference to any other.

For example: (*i*) When a card is drawn from a well shuffled pack, any card may appear in the draw so that the 52 different cases are equally likely.

(*ii*) In throwing a die, all the six faces are equally likely to come.

(*f*) **Independent and dependent events.** Two or more events are said to be *independent* if the happening or non-happening of any one does not depend (or is not affected) by the happening or non-happening of any other. Otherwise they are said to be *dependent*.

For example. If a card is drawn from a pack of well shuffled cards and replaced before drawing the second card, the result of the second draw is independent of the first draw. However, if the first card drawn is not replaced, then, the second draw is dependent on the first draw.

5.2. MATHEMATICAL (or Classical) DEFINITION OF PROBABILITY

If a trial results in n *exhaustive, mutually exclusive* and *equally likely cases* and m of them are favourable to the happening of an event E, then the probability of happening of E is given by

$$p \text{ or } P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}.$$

Note 1. Since the number of cases favourable to happening of E is m and the exhaustive number of cases in n, therefore, the number of cases unfavourable to happening of E are $n - m$.

Note 2. The probability that the event E will not happen is given by

$$q \text{ or } P(\overline{E}) = \frac{\text{Unfavourable number of cases}}{\text{Exhaustive number of cases}} = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

$$p + q = 1 \quad i.e., \quad P(E) + P(\overline{E}) = 1$$

Obviously, p and q are non-negative and cannot exceed unity, *i.e.*, $0 \le p \le 1, 0 \le q \le 1$.

Note 3. If $P(E) = 1$, E is called a *certain event i.e.*, the chance of its happening is cent per cent. If $P(E) = 0$, then E is an *impossible event*.

Note 4. If n cases are favourable to E and m cases are favourable to $\overline{E}$ (*i.e.*, unfavourable to E), then exhaustive number of cases $= n + m$.

$$P(E) = \frac{n}{n+m} \quad \text{and} \quad P(\overline{E}) = \frac{m}{n+m}$$

We say that "odds in *favour of E*" are $n : m$ and "odds *against* E" are $m : n$.

ILLUSTRATIVE EXAMPLES

Example 1. *A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both be white.*

Sol. Total number of balls $= 7 + 6 + 5 = 18$.

Out of 18 balls, 2 can be drawn in ${}^{18}C_2$ ways.

$$\therefore \quad \text{Exhaustive number of cases} = {}^{18}C_2 = \frac{18 \times 17}{2 \times 1} = 153$$

Out of 7 white balls, 2 can be drawn in ${}^7C_2 = \frac{7 \times 6}{2 \times 1} = 21$ ways.

$\therefore$ Favourable number of cases $= 21$

Required probability $= \frac{21}{153} = \frac{7}{51}$.

Example 2. *Four cards are drawn from a pack of cards. Find the probability that (i) all are diamonds, (ii) there is one card of each suit, and (iii) there are two spades and two hearts.*

Sol. 4 cards can be drawn from a pack of 52 cards in $^{52}C_4$ ways.

$\therefore$ Exhaustive number of cases $= {}^{52}C_4 = \dfrac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1} = 270725.$

(*i*) There are 13 diamonds in the pack and 4 can be drawn out of them in $^{13}C_4$ ways.

$\therefore$ Favourable number of cases $= {}^{13}C_4 = \dfrac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 715.$

Required probability $= \dfrac{715}{270725} = \dfrac{143}{54145} = \dfrac{11}{4165}.$

(*ii*) There are 4 suits, each containing 13 cards.

$\therefore$ Favourable number of cases $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13 \times 13 \times 13 \times 13.$

Required probability $= \dfrac{13 \times 13 \times 13 \times 13}{270725} = \dfrac{2197}{20825}.$

(*iii*) 2 spades out of 13 can be drawn in $^{13}C_2$ ways.

2 hearts out of 13 can be drawn in $^{13}C_2$ ways.

$\therefore$ Favourable number of cases $= {}^{13}C_2 \times {}^{13}C_2 = 78 \times 78$

Required probability $= \dfrac{78 \times 78}{270725} = \dfrac{468}{20825}.$

Example 3. *A bag contains 50 tickets numbered 1, 2, 3, ..., 50, of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$). What is the probability that $x_3 = 30$?*

Sol. Exhaustive number of cases $^{50}C_5$.

If $x_3 = 30$, then the two tickets with numbers x_1 and x_2 must come out of 29 tickets numbered 1 to 29 and this can be done in $^{29}C_2$ ways. The other two tickets with numbers x_4 and x_5 must come out of the 20 tickets number 31 to 50 and this can be done in $^{20}C_2$ ways.

$\therefore$ Favourable number of cases $= {}^{29}C_2 \times {}^{20}C_2.$

Required probability $= \dfrac{{}^{29}C_2 \times {}^{20}C_2}{{}^{50}C_5} = \dfrac{551}{15134}.$

5.3. RANDOM EXPERIMENT

Occurrences which can be repeated a number of times, essentially under the same conditions, and whose result cannot be predicted before hand are known as **random experiments.**

For example, rolling of a die, tossing a coin, taking out balls from an urn.

Sample Space. Out of the several possible outcomes of a random experiment, one and only one can take place in a trial. The set of all these possible outcomes is called the **sample space** for the particular experiment and is denoted by S.

For example, if a coin is tossed, the possible outcomes are H (Head) and T (Tail).

Thus S = {H, T}.

Sample Point. The elements of S, the sample space, are called **sample points.**

For example, if a coin is tossed and H and T denote 'Head' and 'Tail' respectively, then S = {H, T}.

The two sample points are H and T.

Finite Sample Space. If the number of sample points in a sample space is finite, we call it a **finite sample space.** (In this chapter, we shall deal with finite sample spaces only).

Event. Every subset of S, the sample space, is called an event.

Since $S \subset S$, S itself is an event; called a **certain event.**

Also, $\phi \subset S$, the null set is also an event, called an **impossible event.**

If $e \in S$, then e is called an **elementary event.** Every elementary event contains only one sample point.

5.4. AXIOMS

(*i*) With each event E (*i.e.*, a sample point) is associated a real number between 0 and 1, called the probability of that event and is denoted by P(E). Thus $0 \le P(E) \le 1$.

(*ii*) The sum of the probabilities of all simple (elementary) events constituting the sample space is 1. Thus P(S) = 1.

(*iii*) The probability of a compound event (*i.e.*, an event made up of two or more sample events) is the sum of the probabilities of the simple events comprising the compound event.

Thus, if there are *n equally likely* possible outcomes of a random experiment, then the sample space S contains *n* sample points and the probability associated with each sample point is $\dfrac{1}{n}$. [By Axiom (*ii*)]

Now, if an event E consists of *m* sample points, then the probability of E is

$$P(E) = \frac{1}{n} + \frac{1}{n} + \text{......} \; m \text{ times} = \frac{m}{n}$$

$$= \frac{\text{Number of sample points in E}}{\text{Number of sample points in S}}.$$

This closely agrees with the classical definition of probability.

5.5. PROBABILITY OF THE IMPOSSIBLE EVENT IS ZERO, *i.e.*, P (ϕ) = 0

Proof. Impossible event contains no sample point. As such, the sample space S and the impossible event ϕ are *mutually exclusive*.

$\Rightarrow \quad S \cup \phi = S \quad \Rightarrow \quad P(S \cup \phi) = P(S)$

$\Rightarrow \quad P(S) + P(\phi) = P(S) \quad \Rightarrow \quad P(\phi) = 0.$

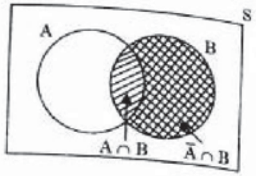
5.6. PROBABILITY OF THE COMPLEMENTARY EVENT $\overline{A}$ or A^c OF A IS GIVEN BY P ($\overline{A}$) = 1 – P(A)

Proof. $\overline{A}$ and A are disjoint events. Also $A \cup \overline{A} = S$

$\therefore \quad P(A \cup \overline{A}) = P(S)$

$\Rightarrow \quad P(A) + P(\overline{A}) = 1$ Hence $P(\overline{A}) = 1 - P(A).$

5.7. FOR ANY TWO EVENTS A AND B, $P(\bar{A} \cap B) = P(B) - P(A \cap B)$



Proof. $\bar{A} \cap B = \{p : p \in B \text{ and } p \notin A\}$

Now $\bar{A} \cap B$ and $A \cap B$ are disjoint sets and

$$(\bar{A} \cap B) \cup (A \cap B) = B$$

$\Rightarrow$ $P[(\bar{A} \cap B) \cup (A \cap B)] = P(B)$

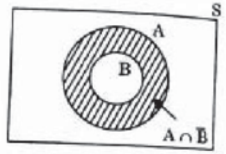
$\Rightarrow$ $P(\bar{A} \cap B) + P(A \cap B) = P(B)$

$\Rightarrow$ $P(\bar{A} \cap B) = P(B) - P(A \cap B)$.

Note. Similarly, it can be proved that $P(A \cap \bar{B}) = P(A) - P(A \cap B)$.

5.8. IF $B \subset A$, THEN (*i*) $P(A \cap \bar{B}) = P(A) - P(B)$ (*ii*) $P(B) \le P(A)$

Proof. When $B \subset A$, B and $A \cap \bar{B}$ are disjoint and their union is A.



$\Rightarrow$ $B \cup (A \cap \bar{B}) = A$

$\Rightarrow$ $P[B \cup (A \cap \bar{B})] = P(A)$

$\Rightarrow$ $P(B) + P(A \cap \bar{B}) = P(A)$

$\Rightarrow$ $P(A \cap \bar{B}) = P(A) - P(B)$ ...(1)

Now, if E is any event,

then $0 \le P(E) \le 1$, *i.e.*, $P(E) \ge 0$

$\therefore$ $P(A \cap \bar{B}) \ge 0 \Rightarrow P(A) - P(B) \ge 0$ [Using 1]

$\Rightarrow$ $P(B) \le P(A)$.

5.9. $P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$

Proof. By 5.12 $B \subset A \Rightarrow P(B) \le P(A)$

Since $(A \cap B) \subset A$ and $(A \cap B) \subset B$

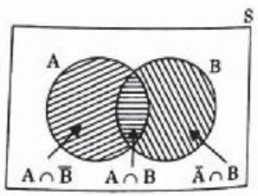
$\therefore$ $P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$.

5.10. ADDITION THEOREM OF PROBABILITY

Statement. *If A and B are any two events, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e., $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.



Proof. A and $\bar{A} \cap B$ are *disjoint sets* and their union is $A \cup B$.

$\Rightarrow$ $A \cup B = A \cup (\bar{A} \cap B)$

$\Rightarrow$ $P(A \cup B) = P[(A \cup (\bar{A} \cap B)] = P(A) + P(\bar{A} \cap B)$

$= P(A) + [P(\bar{A} \cap B) + P(A \cap B) - P(A \cap B)]$

$= P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B)$ [$\because$ $\bar{A} \cap B$ and $A \cap B$ are disjoint]

$= P(A) + P(B) - P(A \cap B)$ [$\because$ $(\bar{A} \cap B) \cup (A \cap B) = B$]

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Note 1. If A and B are two mutually disjoint events, then $A \cap B = \phi$, so that $P(A \cap B) = P(\phi) = 0$.

$\therefore$ $P(A \cup B) = P(A) + P(B)$.

Note 2. $P(A \cup B)$ is also written as $P(A + B)$. Thus, for mutually disjoint events A and B,

$$P(A + B) = P(A) + P(B).$$

$P(A \cap B)$ is also written as $P(AB)$.

5.11. IF A, B AND C ARE ANY THREE EVENTS, THEN

$$\mathbf{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)}$$

Or

$$\mathbf{P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC).}$$

Proof. Using the above Art. 5.10 for two events, we have

$P(A \cup B \cup C) = P[(A \cup B) \cup C]$

$= P(A \cup B) + P(C) - P[(A \cup B) \cap C]$

$= [P(A) + P(B) - P(A \cap B)] + P(C) - P[(A \cap C) \cup (B \cap C)]$ [By distributive law]

$= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C)]$
$+ P(B \cap C) - P[(A \cap C) \cap (B \cap C)]$ [By 5.10]

$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
[$\because$ $(A \cap C) \cap (B \cap C) = A \cap B \cap C$]

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
[$\because$ $A \cap C = C \cap A$]

or $P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$.

5.12. IF $A_1, A_2, \ldots, A_n$ ARE n MUTUALLY EXCLUSIVE EVENTS, THEN THE PROBABILITY OF THE HAPPENING OF ONE OF THEM IS $P(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) = P(A_1 + A_2 + \ldots + A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Proof. Let N be the total number of mutually exclusive, exhaustive and equally likely cases of which m_1 are favourable to A_1, m_2 are favourable to A_2 and so on.

$$\left.\begin{aligned} &\text{Probability of occurrence of event } A_1 = P(A_1) = \frac{m_1}{N} \\ &\text{Probability of occurrence of event } A_2 = P(A_2) = \frac{m_2}{N} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\text{Probability of occurrence of event } A_n = P(A_n) = \frac{m_n}{N} \end{aligned}\right\} \quad \ldots(1)$$

The events being mutually exclusive and equally likely, the number of cases favourable to the event

$$A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_n \text{ is } m_1 + m_2 + \ldots + m_n.$$

$\therefore$ Probability of occurrence of one of the events $A_1, A_2, \ldots\ldots, A_n$ is $P(A_1 + A_2 + \ldots + A_n)$

$$= \frac{m_1 + m_2 + \ldots + m_n}{N} = \frac{m_1}{N} + \frac{m_2}{N} + \ldots\ldots + \frac{m_n}{N}$$

$= P(A_1) + P(A_2) + \ldots\ldots + P(A_n)$ | Using (1)

Note. The student should not get confused with Theorems 5.10, 5.11 and 5.12. Theorems 5.10 and 5.11 are for ANY events (mutually exclusive or not) whereas Theorem 5.11 is for mutually exclusive events.

ILLUSTRATIVE EXAMPLES

Example 1. *In a given race, the odds in favour of four horses A, B, C, D are 1 : 3, 1 : 4, 1 : 5, 1 : 6 respectively. Assuming that a dead heat is impossible; find the chance that one of them wins the race.*

Sol. Let p_1, p_2, p_3, p_4 be the probabilities of winning of the horses A, B, C, D respectively.

Since a *dead heat* (in which all the four horses cover *same distance in same time*) is not possible, the events are mutually exclusive.

Odds in favour of A are 1 : 3 $\therefore\ p_1 = \frac{1}{1+3} = \frac{1}{4}$

Similarly, $p_2 = \frac{1}{5}, p_3 = \frac{1}{6}, p_4 = \frac{1}{7}$.

If p is the chance that one of them wins, then

$$p = p_1 + p_2 + p_3 + p_4 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{319}{420}.$$

Example 2. *A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace ?* AUE

Sol. Let A = the event of drawing a spade

and B = the event of drawing an ace

A and B are *not* mutually exclusive.

AB = the even of drawing the ace of spades

$$P(A) = \frac{13}{52}, P(B) = \frac{4}{52}, P(A \cap B) = \frac{1}{52}$$

$$\therefore \quad P(A \text{ or } B) = P(A) + P(B) - P(A \cap B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

5.13. CONDITIONAL PROBABILITY

The probability of the happening of an event E_1 when another event E_2 is known to have already happened is called *Conditional Probability* and is denoted by $P(E_1/E_2)$.

Mutually Independent Events. An event E_1 is said to be independent of an event E_2 if

$$P(E_1/E_2) = P(E_1).$$

i.e., if the probability of happening of E_1 is independent of the happening of E_2.

5.14. MULTIPLICATIVE LAW OF PROBABILITY (Or Theorem of Compound Probability)

The probability of simultaneous occurrence of two events is equal to the probability of one of the events multiplied by the conditional probability of the other, *i.e.*, **for two events A and B,**

$$P(A \cap B) = P(A) \times P(B/A)$$

where P(B/A) represents the conditional probability of occurrence of B when the event A has already happened.

Note. $P(A \cap B)$ is also written as P(AB).

Thus $P(AB) = P(A) \times P(B/A)$.

Cor. 1. Interchanging A and B

$$P(BA) = P(B) \times P(A/B)$$

or $$P(AB) = P(B) \times P(A/B) \qquad [\because\ B \cap A = A \cap B]$$

Cor. 2. If A and B are independent events, then P(B/A) = P(B)

$$\therefore \quad P(AB) = P(A) \times P(B).$$

Generalisation. If $A_1, A_2, \ldots\ldots, A_n$ are n independent events, then

$$P(A_1 A_2 \ldots A_n) = P(A_1) \times P(A_2) \times \ldots\ldots \times P(A_n).$$

Cor. 3. If p is the chance that an event will happen in one trial then the chance that it will happen in a succession of r trials is

$$p.p \ldots\ldots p\ .\ (r \text{ times}) = p^r.$$

Cor. 4. If $p_1, p_2, \ldots\ldots, p_n$ are the probabilities that certain events happen, then the probabilities of their non-happening are $1 - p_1, 1 - p_2, \ldots\ldots, 1 - p_n$ and, therefore, the probability of all of these failing is

$$(1 - p_1)(1 - p_2)\ldots\ldots(1 - p_n).$$

Hence the chance in which at least one of these events much happen is

$$1 - (1 - p_1)(1 - p_2) \ldots\ldots (1 - p_n).$$

Cor. 5. $P(A/B) = \frac{P(A \cap B)}{P(A)}, \quad P(B/A) = \frac{P(A \cap B)}{P(A)}$

Cor. 6. Theorem of Total Probability

If events E_1, E_2, E_3 are mutually exclusive and exhaustive, then for any event A, we have

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)$$

ILLUSTRATIVE EXAMPLES

Example 1. *A problem in mechanics is given to three students A, B, C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. What is the probability that the problem will be solved ?*

Sol. The probabilities of A, B, C solving the problem are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

The probabilities of A, B, C not solving the problem are $1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}$ *i.e.*, $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

$\therefore$ The probability that the problem is not solved by any of them $= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$.

Hence the probability that the problem is solved by at least one of them $= 1 - \frac{1}{4} = \frac{3}{4}$.

Example 2. *The odds that a book will be favourably reviewed by three independent critics are 5 to 2, 4 to 3 and 3 to 4 respectively. What is the probability that, of the three reviews, a majority will be favourable ?*

Sol. Let the three critics be A, B, C. The probabilities p_1, p_2, p_3 of the book being favourably reviewed by A, B, C are $\frac{5}{7}, \frac{4}{7}, \frac{3}{7}$ respectively.

$\therefore$ The probabilities that the book is unfavourably reviewed by A, B, C are

$$1 - \frac{5}{7} = \frac{2}{7},\ 1 - \frac{4}{7} = \frac{3}{7},\ 1 - \frac{3}{7} = \frac{4}{7}.$$

A majority will be favourable if the reviews of at least two are favourable.

(*i*) If A, B, C all review favourably, the probability is

$$\frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{60}{343} \qquad | \ p_1p_2p_3$$

(*ii*) If A, B review favourably and C reviews unfavourably, the probability is

$$\frac{5}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{80}{343} \qquad | \ p_1p_2(1 - p_3)$$

(*iii*) If A, C review favourably and B reviews unfavourably, probability is

$$\frac{5}{7} \times \frac{3}{7} \times \frac{3}{7} = \frac{45}{343} \qquad | \ p_1(1 - p_2)p_3$$

(*iv*) If B, C review favourably and A reviews unfavourably, the probability is

$$\frac{2}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{24}{343} \qquad | \ (1 - p_1)p_3p_2$$

Hence the probability that a majority will be favourable is

$$\frac{60}{343} + \frac{80}{343} + \frac{45}{343} + \frac{24}{343} = \frac{209}{343}.$$

Example 3. *A can hit a target 4 times in 5 shots; B 3 times 4 shots; C twice in 3 shots. They fire a volley. What is the probability that at least two shots hit ?*

Sol. Probability of A's hitting the target $= \frac{4}{5}$

Probability of B's hitting the target $= \frac{3}{4}$

Probability of C's hitting the target $= \frac{2}{3}$.

For at least two hits, we may have

(*i*) A, B, C all hit the target, the probability for which is

$$\frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{24}{60}.$$

(*ii*) A, B hit the target and C misses it, the probability for which is

$$\frac{4}{5} \times \frac{3}{4} \times \left(1 - \frac{2}{3}\right) = \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{12}{60}.$$

(*iii*) A, C hit the target and B misses it, the probability for which is

$$\frac{4}{5} \times \left(1 - \frac{3}{4}\right) \times \frac{2}{3} = \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} = \frac{8}{60}.$$

(*iv*) B, C hit the target and A misses it, the probability for which is

$$\left(1 - \frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{6}{60}.$$

Since these are mutually exclusive events, required probability

$$= \frac{24}{60} + \frac{12}{60} + \frac{8}{60} + \frac{6}{60} = \frac{50}{60} = \frac{5}{6}.$$

Example 4. *A has 2 shares in a lottery in which there are 3 prizes and 5 blanks; B has 3 shares in a lottery in which there are 4 prizes and 6 blanks. Show that A's chance of success is to B's as 27 : 35.*

Sol. A can draw two tickets (out of 3 + 5 = 8) in ${}^8C_2 = 28$ ways.

A will get all blanks in ${}^5C_2 = 10$ ways. $\therefore$ A can win a prize in 28 – 10 = 18 ways

Hence A's chance of success $= \frac{18}{28} = \frac{9}{14}$

B can draw 3 tickets (out of 4 + 6 = 10) in ${}^{10}C_3 = 120$ ways; B will get all blanks in ${}^6C_3 = 20$ ways.

$\therefore$ B can win a prize in 120 – 20 = 100 ways.

Hence B's chance of success $= \frac{100}{120} = \frac{5}{6}$.

$\therefore$ A's chance : B's chance $= \frac{9}{14} : \frac{5}{6} = 27 : 35$.

Example 5. *A and B throw alternately with a single die, A having the first throw. The person who first throws ace is to win. What are their respective chances of winning ?*

Sol. The chance of throwing an ace with a single die $= \frac{1}{6}$

The chance of not throwing an ace with a single die $= 1 - \frac{1}{6} = \frac{5}{6}$.

If A is to win, he should throw an ace in the first or third or fifth, ..., throws.

If B is to win, he should throw an ace in the second or fourth or sixth, ..., throws.

The chances that an ace is thrown in the first, second, third, ..., throws are

$$\frac{1}{6}, \frac{5}{6}\cdot\frac{1}{6}, \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}, \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} \ \ldots\ldots \quad \text{or} \quad \frac{1}{6}, \frac{5}{6}\cdot\frac{1}{6}, \left(\frac{5}{6}\right)^2\cdot\frac{1}{6}, \left(\frac{5}{6}\right)^3\cdot\frac{1}{6}, \ldots\ldots$$

$$\therefore \text{ A's chance} = \frac{1}{6} + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^4\cdot\frac{1}{6} + \ldots\ldots = \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{6}{11} \qquad \left| \text{ Sum of an infinite G.P.} = \frac{a}{1-r}\right.$$

$$\text{B's chance} = 1 - \frac{6}{11} = \frac{5}{11}.$$

Example 6. *Cards are dealt one by one from a well-shuffled pack until an ace appears. Show that the probability that exactly n cards are dealt before the first ace appears is*

$$\frac{4(51-n)(50-n)(49-n)}{52.51.50.49}.$$

Sol. Let A be the event of drawing n non-ace cards and B, the event of drawing an ace in the $(n+1)$th draw.

Consider the event A

n cards can be drawn out of 52 cards in $^{52}C_n$ ways.

$\Rightarrow$ Exhaustive cases $= {}^{52}C_n$

n non-ace cards can be drawn out of 52 cards in $^{48}C_n$ ways.

$\Rightarrow$ Favourable cases $= {}^{48}C_n$

$$\therefore \quad P(A) = {}^{48}C_n/{}^{52}C_n = \frac{48!}{(48-n)!\,n!} \times \frac{(52-n)!\,(n)!}{52!}$$

$$= \frac{48!\,.(52-n)(51-n)(50-n)(49-n)(48-n)!}{(48-n)!\,.52.51.50.49.(48)!} = \frac{(52-n)(51-n)(50-n)(49-n)}{52.51.50.49}.$$

Consider the event B

n cards have already been drawn in the first n draws.

Exhaustive cases $= {}^{52-n}C_1 = 52-n$; Favourable cases $= {}^4C_1 = 4$

$$\therefore \quad P(B/A) = \frac{4}{52-n}$$

Reqd. Probability $= P(A)\,.\,P(B/A)$

$$= \frac{(52-n)(51-n)(50-n)(49-n)}{52.51.50.49} \times \frac{4}{52-n} = \frac{4(51-n)(50-n)(49-n)}{52.51.50.49}.$$

Example 7. *An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the latter. What is the probability that it is a white ball ?*

Sol. The two balls drawn from the first urn may be

(*i*) both white (*ii*) both black (*iii*) one white and one black.

Let these events be denoted by A, B, C respectively.

$$P(A) = \frac{^{10}C_2}{^{13}C_2} = \frac{10 \times 9}{13 \times 12} = \frac{15}{26}; \qquad P(B) = \frac{^{3}C_2}{^{13}C_2} = \frac{3 \times 2}{13 \times 12} = \frac{1}{26}$$

$$P(C) = \frac{^{10}C_1 \times {}^3C_1}{^{13}C_2} = \frac{10 \times 3}{\frac{13 \times 12}{2 \times 1}} = \frac{10}{26}$$

When two balls are transferred from first urn to second urn, the second urn will contain.

(*i*) 5 white and 5 black balls (*ii*) 3 white and 7 black balls

(*iii*) 4 white and 6 black balls.

Let W denote the event of drawing a white ball from the second urn in the three cases (*i*), (*ii*) and (*iii*).

Now $\quad P(W/A) = \dfrac{5}{10}, \quad P(W/B) = \dfrac{3}{10}, \quad P(W/C) = \dfrac{4}{10}$

$\therefore$ Reqd. probability $= P(A)\,.\,P(W/A) + P(B)\,.\,P(W/B) + P(C)\,.\,P(W/C)$

$$= \frac{15}{26}\cdot\frac{5}{10} + \frac{1}{26}\cdot\frac{3}{10} + \frac{10}{26}\cdot\frac{4}{10} = \frac{75+3+40}{260} = \frac{118}{260} = \frac{59}{130}.$$

Example 8. *Three bags A, B, C contain 4 red, 3 black, 2 white; 3 red, 4 black, 4 white; and 5 red, 2 black, 6 white balls respectively. If a bag is selected at random and a ball is drawn from it, find the probability that the ball drawn is red.* (M.D.U. May 2006, Dec. 2007)

Sol. Let E_1, E_2, E_3 denote the events of choosing bags A, B, C respectively,

then $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Let R denote the event of drawing a red ball, then

$$P(R/E_1) = \frac{4}{4+3+2} = \frac{4}{9}, \quad P(R/E_2) = \frac{3}{3+4+4} = \frac{3}{11},$$

$$P(R/E_3) = \frac{5}{5+2+6} = \frac{5}{13}$$

$$\therefore \quad P(R) = P(E_1)\,P(R/E_1) + P(E_2)\,P(R/E_2) + P(E_3)\,P(R/E_3)$$

$$= \frac{1}{3}\times\frac{4}{9} + \frac{1}{3}\times\frac{3}{11} + \frac{1}{3}\times\frac{5}{13} = \frac{4}{27} + \frac{1}{11} + \frac{5}{39}$$

$$= 0.148 + 0.090 + 0.128 = 0.366.$$

EXERCISE 5.1

1. When a coin is tossed four times, find the probability of getting (*i*) exactly one head, (*ii*) at most three heads and (*iii*) at least two heads ?
2. What is the chance that a (*i*) non-leap year (*ii*) leap year should have fifty three Sundays ? (*P.T.U. May 2006*)
3. A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning the bet ?
4. An integer is chosen at random from the first two hundred positive integers. What is the probability that the integer chosen is divisible by 6 or 8 ?
5. Six cards are drawn at random from a pack of 52 cards. What is the probability that 3 will be red and 3 black ?
6. (*a*) From a set of raffle tickets numbered 1 to 100, three are drawn at random. What is the probability that all are odd numbered ?

   (*b*) A box contains 9 tickets numbered 1 to 9 inclusive. If 3 tickets are drawn from the box, one at a time, find the probability they are alternatively either odd, even, odd or even, odd, even. (*M.D.U. Dec. 2009*)
7. (*a*) If from a lottery of 30 tickets, marked, 1, 2, 3, ..., 30, four tickets be drawn, what is the chance that those marked 1 and 2 are among them ?

   (*b*) An urn contains 5 red and 10 black balls. Eight of them are placed in another urn. What is the chance that the latter then contains 2 red and 6 black balls ?
8. A party of n persons sit at a round table. Find the odds against two specified individuals sitting next to each other.
9. A five-figured number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the probability that the number formed is divisible by 4.

10. Three newspapers A, B, C are published in a city and a survey of readers indicates the following :
20% read A, 16% read B, 14% read C,
8% read both A and B, 5% read both A and C,
4% read both B and C, 2% read all the three.
For a person chosen at random, find the probability that he reads none of the papers.

11. A problem is statistics is given to five students. Their chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}$ and $\frac{1}{5}$. What is the probability that the problem will be solved ?

12. A can hit a target 3 times in 5 shots, B 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that (*i*) two shots hit, (*ii*) at least two shots hit the target ?

13. Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected consist of 1 girl and 2 boys is $\frac{13}{32}$.

14. Suppose that 5 men out of 100 and 25 women out of 1000 are good orators. Assuming that there are equal number of men and women, find the probability of choosing a good orator.

15. A bag contains 10 balls, two of which are red, three blue and five black. Three balls are drawn at random from the bag. What is the probability that
(*i*) the three balls are of different colours, (*ii*) two balls are of the same colour,
(*iii*) the balls are all of the same colour.

16. It is 8 : 5 against a person who is 40 years old living till he is 70 and 4 : 3 against a person now 50 living till he is 80. Find the probability that one at least of these persons will be alive 30 years hence.

17. An old carton in a pharmacy shop is found to contain 60 capsules and 180 tablets. Half of capsules and tablets are of expiry date. Find the probability that an item picked at random from the carton is of expiry date or a capsule.

18. A has 3 shares in a lottery where there are 3 prizes and 6 blanks. B has one share in another, where there is just one prize and two blanks. Show that A has a better chance of winning a prize than B in the ratio 16 : 7.

19. A, B and C, in order, toss a coin. The first one to throw a head wins. If A starts, find their respective chances of winning.

20. (*a*) A speaks truth in 75% cases and B in 80% cases. In what percentages of cases are they likely to contradict each other in stating the same fact ?
(*b*) A pair of dice is tossed twice. Find the probability of scoring 7 points (*i*) once, (*ii*) at least once (*iii*) twice. (*K.U.K. Dec. 2009*)

21. (*a*) Two persons A and B toss an unbiased coin alternately on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning.
(*b*) A and B throw alternately with a pair of ordinary dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find their respective chances of winning. (Huyghen's Problem) (*Madras 2006*)

22. (*a*) Two cards are randomly drawn from a deck of 52 cards and thrown away. What is the probability of drawing an ace in a single draw from the remaining 50 cards ?
(*b*) A box A contains 2 white and 4 black balls. Another box B contains 5 white and 7 black balls. A ball is transferred from the box A to the box B, then a ball is drawn from the box B. Find the probability that it is white.

23. A committee consists of 9 students two of which are from 1st year, three from 2nd year and four from 3rd year. Three students are to be removed at random. What is the chance that
(*i*) the three students belong to different classes,

(*ii*) two belong to the same class and third to the different class, and
(*iii*) the three belong to the same class ?

24. Five men in a company of twenty are graduates. If 3 men and picked out of 20 at random, what is the probability that
(*i*) they are all graduates ? (*ii*) at least one is graduate ?

25. If A, B, C are events such that
$P(A) = 0.3, P(B) = 0.4, P(C) = 0.8$
$P(A \cap B) = 0.08, P(A \cap C) = 0.28, P(A \cap B \cap C) = 0.09$
If $P(A \cup B \cup C) \geq 0.75$, then show that $0.23 \leq P(B \cap C) \leq 0.48$.

26. For two events A and B, let $P(A) = 0.4$, $P(B) = p$ and $P(A \cup B) = 0.6$
(*i*) Find p so that A and B are independent events.
(*ii*) For what value of p are A and B are mutually exclusive ?

27. A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that
(*i*) both of them will be selected, (*ii*) only one of them will be selected, and
(*iii*) none of them will be selected ?

28. Two dice are tossed once. Find the probability of getting an even number on the first die or a total of 8.

29. There are two bags. The first bag contains 5 white and 3 black balls and the second bag contains 3 white and 5 black balls. Two balls are drawn at random from the first bag and without noting their colours, put into the second bag. Then two balls are drawn from the second bag. Find the probability that the balls drawn are white and black.

30. A purse contains 2 silver and 4 copper coins. A second purse contains 4 silver and 3 copper coins. If a coin is pulled out at random from one of the two purses, what is the probability that it is a silver coin ?

31. A man wants to marry a girl having qualities: white complexion—the probability of getting such a girl is one in twenty; handsome dowry—the probability of getting this is one in fifty ; westernised manners and etiquettes—the probability here is one in hundred. Find out the probability of his getting married to such a girl when the possession of these three attributes is independent.

32. A class consists of 80 students, 25 of them are girls and 55 boys, 10 of them are rich and the remaining poor, 20 of them are fair complexioned. What is the probability of selecting a fair complexioned rich girl ?

33. Of the students attending a lecture, 50% could not see what was written on the board and 40% could not hear what the lecturer was saying. Most unfortunate 30% fell into both of these categories. What is the probability that a student picked at random was able to see and hear satisfactorily ?

34. The probabilities of A, B, C solving a problem are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

35. A student takes his examination in four subjects $\alpha, \beta, \gamma, \delta$. He estimates his chance of passing in α as $\frac{4}{5}$, in β as $\frac{3}{4}$, in γ as $\frac{5}{6}$ and in δ as $\frac{2}{3}$. To qualify he must pass in α and at least two other subjects. What is the probability that the qualifies ?

36. For any two events A and B, prove that
$$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B).$$

37. A student takes his examination in four subjects P, Q, R, S. He estimates his chances of passing in P as $\frac{4}{5}$, in Q as $\frac{3}{4}$, in R as $\frac{5}{6}$ and in S as $\frac{2}{3}$. To qualify, he must pass in P and at least two other subjects. What is the probability that he qualifies ?

38. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number ?

39. A die is thrown three times. Events A and B are defined as below:

A: 4 appears on third throw

B: 6 and 5 appear respectively on first two throws

Find the probability of A given that B has already occurred.

40. A fair die is rolled. If E = {1, 3, 5}, F = {2, 3} and G = {2, 3, 4, 5}, find P ((E $\cup$ F)/G) and P ((E $\cap$ F)/G).

41. A person has undertaken a construction job. The probabilities are 0.65 that there will be a strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

Answers

1. $\frac{1}{4}, \frac{15}{16}, \frac{11}{16}$ **2.** (*i*) $\frac{1}{7}$ (*ii*) $\frac{2}{7}$ **3.** 9: 4 **4.** $\frac{1}{4}$

5. $\frac{13000}{39151}$ **6.** (*a*) $\frac{4}{33}$ (*b*) $\frac{5}{18}$ **7.** (*a*) $\frac{2}{145}$, (*b*) $\frac{140}{429}$

8. $(n-3)$: 2 **9.** $\frac{5}{16}$ **10.** $\frac{13}{20}$ **11.** $\frac{17}{20}$

12. 0.45, 0.63 **14.** 0.0375 **15.** (*i*) $\frac{1}{4}$ (*ii*) $\frac{79}{120}$ (*iii*) $\frac{11}{120}$

16. $\frac{59}{91}$ **17.** $\frac{5}{8}$ **19.** $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$

20. (*a*) 35% (*b*) (*i*) $\frac{5}{18}$ (*ii*) $\frac{11}{36}$ (*iii*) $\frac{1}{36}$ **21.** (*a*) $\frac{2}{3}, \frac{1}{3}$ (*b*) $\frac{30}{61}, \frac{31}{61}$

22. (*a*) $\frac{1}{13}$, (*b*) $\frac{16}{39}$ **23.** (*i*) $\frac{2}{7}$ (*ii*) $\frac{55}{84}$ (*iii*) $\frac{5}{84}$ **24.** (*i*) $\frac{1}{114}$ (*ii*) $\frac{137}{228}$

26. (*i*) $\frac{1}{3}$ (*ii*) 0.2 **27.** (*i*) $\frac{1}{35}$ (*ii*) $\frac{2}{7}$ (*iii*) $\frac{24}{35}$ **28.** $\frac{5}{9}$

29. $\frac{673}{1260}$ **30.** $\frac{19}{42}$ **31.** 0.000001 **32.** $\frac{5}{512}$

33. $\frac{2}{5}$ **34.** $\frac{25}{56}$ **35.** $\frac{61}{90}$ **37.** $\frac{61}{90}$

38. $\frac{4}{7}$ **39.** $\frac{1}{6}$ **40.** $\frac{3}{4}, \frac{1}{4}$ **41.** 0.488.

5.15. BAYE'S THEOREM

(M.D.U. Dec. 2010)

If $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0$, (i = 1, 2, ..., n) of a random experiment then for any arbitrary event A of the sample space of the above experiment with $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)}.$$

Proof. Let S be the sample space of the random experiment.

The events $E_1, E_2, \ldots\ldots, E_n$ being exhaustive

$$S = E_1 \cup E_2 \cup \ldots\ldots \cup E_n$$

$$\therefore \quad A = A \cap S \qquad [\because A \subset S]$$

$$= A \cap (E_1 \cup E_2 \cup \ldots\ldots \cup E_n)$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup \ldots\ldots \cup (A \cap E_n) \qquad \text{[Distributive Law]}$$

$$\Rightarrow \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots\ldots + P(A \cap E_n)$$

$$= P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + \ldots\ldots + P(E_n)P(A/E_n)$$

$$= \sum_{i=1}^{n} P(E_i)P(A/E_i) \qquad \ldots(1)$$

Now $\quad P(A \cap E_i) = P(A)P(E_i/A)$

$$\Rightarrow \quad P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)} \qquad \text{[Using (1)]}$$

Note. The significance of Baye's Theorem may be understood in the following manner:

$P(E_i)$ is the probability of occurrence of E_i. The experiment is performed and we are told that the event A has occurred. With this information, the probability $P(E_i)$ is changed to $P(E_i/A)$. Baye's Theorem enables us to evaluate $P(E_i/A)$ if all the $P(E_i)$ and the conditional probabilities $P(A/E_i)$ are known.

ILLUSTRATIVE EXAMPLES

Example 1. *A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.*

Sol. Let $\quad E_1$: the ball is drawn from bag X; E_2: the ball is drawn from bag Y

and $\quad$ A: the ball is red.

We have to find $P(E_2/A)$. By Baye's Theorem,

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)} \qquad \ldots(1)$$

Since the two bags are equally likely to be selected, $P(E_1) = P(E_2) = \frac{1}{2}$

Also $\quad P(A/E_1) = P(\text{a red ball is drawn from bag X}) = \frac{3}{5}$

$$P(A/E_2) = P(\text{a red ball is drawn from bag Y}) = \frac{5}{9}$$

$\therefore$ From (1), we have $P(E_2/A) = \dfrac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}} = \dfrac{25}{52}$.

Example 2. *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B ?* (K.U.K. Dec. 2010; V.T.U. 2006)

Sol. Let E_1, E_2 and E_3 denote the events that a bolt selected at random is manufacture by the machines A, B and C respectively and let H denote the event of its being defective. Then

$$P(E_1) = 0.25,\ P(E_2) = 0.35,\ P(E_3) = 0.40$$

The probability of drawing a defective bolt manufactured by machine A is $P(H/E_1) = 0.05$

Similarly, $P(H/E_2) = 0.04$ and $P(H/E_3) = 0.02$

By Baye's Theorem, we have

$$P(E_2/H) = \frac{P(E_2)P(H/E_2)}{P(E_1)P(H/E_1) + P(E_2)P(H/E_2) + P(E_3)P(H/E_3)}$$

$$= \frac{0.35\times 0.04}{0.25\times 0.05 + 0.35\times 0.04 + 0.40\times 0.02} = \frac{0.0140}{0.0345} = 0.41.$$

Example 3. *The contents of urns I, II and III are as follows:*

1 white, 2 black and 3 red balls,

2 white, 1 black and 1 red balls, and

4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III ? (K.U.K. 2005)

Sol. Let E_1: urn I is chosen; E_2: urn II is chosen; E_3: urn III is chosen

and A: the two balls are white and red.

We have to find $P(E_1/A)$, $P(E_2/A)$ and $P(E_3/A)$.

Now $P(E_1) = P(E_2) = P(E_3) = \dfrac{1}{3}$

$$P(A/E_1) = P\ (\text{a white and a red ball are drawn from urn I}) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5}$$

$$P(A/E_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{1}{3};\ P(A/E_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}.$$

By Baye's Theorem, we have

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)} = \frac{\frac{1}{3}\times\frac{1}{5}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{11}} = \frac{33}{118}$$

Similarly $P(E_2/A) = \dfrac{55}{118}$, $P(E_3/A) = \dfrac{15}{59}$.

EXERCISE 5.2

1. (*a*) Two urns contain 4 white, 6 blue and 4 white, 5 blue balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is white, find the probability that it is drawn from the

   (*i*) first urn (*ii*) second urn.

   (*b*) Of the cigarette smoking population, 70% are men and 30% women, 10% of these men and 20% of these women smoke 'WILLS'. What is the probability that a person seen smoking a 'WILLS' will be a man?

2. (*a*) Three urns contain 6 red, 4 black; 4 red, 6 black and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn. (*M.D.U. Dec. 2010*)

   (*b*) There are three bags: first containing 1 white, 2 red, 3 green balls; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. They are found to be 1 red and 1 white. Find the probability that balls so drawn came from the second bag. (*M.D.U. Dec. 2008*)

3. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A were defective and 1% produced by machine B were defective. If a defective item is drawn at random, what is the probability that it was produced by machine A?

4. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of accident is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets an accident. What is the probability that he is a scooter driver? (*M.D.U. 2006*)

5. A company has two plants to manufacture scooters. Plant I manufactures 70% of scooters and plant II manufactures 30%. At plant I, 80% of the scooters are rated standard quality and at plant II, 90% of the scooters are rated standard quality. A scooter is chosen at random and is found to be of standard quality. What is the chance that it has come from plant II?

6. In a bolt factory, there are four machines A, B, C, D manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2%, in the same order, are defective bolts. A bolt is chosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine A or machine D? (*M.D.U., Dec. 2006*)

7. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

8. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

9. A survey was conducted to find the supplies of the consumer durables for the market. It was found that the three major companies A, B and C have market share of 35%, 25% and 40% respectively out of which 2%, 1% and 3% are not upto the satisfaction. A consumer buys a product and is dissatisfied with it. What is the probability that it might be from the company C? *(M.D.U. May 2006)*

10. By examining the chest X-ray, the probability that T.B. is detected when a person is actually suffering is 0.99. The probability that the doctor diagnoses incorrectly that a person has T.B. on the basis of X-ray is 0.001. In a certain city, 1 in 1000 persons suffer from T.B. A person is selected at random and is diagnosed to have T.B. What is the chance that he actually has T.B.?

11. Assume that the chance of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga.

12. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads. What is the probability that it was the two headed coin?

Answers

1. (a) (i) $\frac{9}{19}$ (ii) $\frac{10}{19}$ (b) $\frac{7}{13}$
2. (a) $\frac{2}{5}$ (b) $\frac{6}{11}$
3. $\frac{3}{4}$
4. $\frac{1}{52}$
5. $\frac{27}{83}$
6. 0.3175, 0.254
7. $\frac{1}{2}$
8. $\frac{3}{8}$
9. $\frac{24}{43}$
10. $\frac{110}{221}$
11. $\frac{14}{29}$
12. $\frac{4}{9}$.

5.16. RANDOM VARIABLE

If the numerical values assumed by a variable are the result of some chance factors, so that a particular value cannot be exactly predicted in advance, the variable is then called a *random variable*. A random variable is also called *'chance variable'* or *'stochastic variable'*.

Random variables are denoted by capital letters, usually, from the last part of the alphabet, for instance, X, Y, Z etc.

Continuous and Discrete Random Variables

A *continuous random variable* is one which can assume any value within an interval, *i.e.*, all values of a continuous scale. For example (*i*) the weights (in kg) of a group of individuals, (*ii*) the heights of a group of individuals.

A *discrete random variable* is one which can assume only isolated values. For example,

(*i*) the number of heads in 4 tosses of a coin is a discrete random variable as it cannot assume values other than 0, 1, 2, 3, 4.

(*ii*) the number of aces in a draw of 2 cards from a well shuffled deck is a random variable as it can take the values 0, 1, 2 only.

5.17. (a) DISCRETE PROBABILITY DISTRIBUTION

Let a random variable X assume values $x_1, x_2, x_3, \ldots\ldots, x_n$ with probabilities $p_1, p_2, p_3, \ldots\ldots, p_n$ respectively, where $P(X = x_i) = p_i \ge 0$ for each x_i and $p_1 + p_2 + p_3 + \ldots\ldots + p_n = \sum_{i=1}^{n} p_i = 1$.

Then

$$\begin{array}{lll} X & : & x_1, \quad x_2, \quad x_3, \ldots\ldots, x_n \\ P(X) & : & p_1, \quad p_2, \quad p_3, \ldots\ldots, p_n \end{array}$$

is called the discrete probability distribution for X and it spells out how a total probability of 1 is distributed over several values of the random variable.

5.17. (b) CONTINUOUS PROBABILITY DISTRIBUTION

Let X be a continuous random variable taking values in the interval $(-\infty, \infty)$. Let $f(x)$ be a function satisfying the following properties:

(*i*) $f(x)$ is integrable on $(-\infty, \infty)$

(*ii*) $f(x) \ge 0$ for all x in $(-\infty, \infty)$

(*iii*) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Then $f(x)$ is called the **probability distribution (or density) function** (p.d.f.) of the continuous random variable X.

The probability for a continuous random variable X to fall in the interval $[a, b]$ is

$$P(a \le X \le b) = \int_a^b f(x)\, dx$$

which is nothing but the area between the continuous curve $y = f(x)$, $x = a$, $x = b$ and x-axis. The curve $y = f(x)$ is called the **probability curve.**

Taking $b = a$, $P(X = a) = \int_a^a f(x)\, dx = 0$. Also,

$$P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)$$

If X is a continuous random variable with p.d.f. $f(x)$, then

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx$$

is called **cumulative distribution function** or simply **distribution function** of the continuous random variable X. The distribution function has the following properties:

(*i*) $0 \le F(x) \le 1, \quad -\infty < x < \infty$

(*ii*) $F'(x) = f(x) \ge 0$

(*iii*) $P(a \le X \le b) = \int_a^b f(x)\, dx = \int_{-\infty}^{b} f(x)\, dx - \int_{-\infty}^{a} f(x)\, dx = P(X \le b) - P(X \le a) = F(b) - F(a)$

5.18. MEAN AND VARIANCE OF RANDOM VARIABLES

(a) Let X: $x_1, x_2, x_3, \ldots\ldots, x_n$
P(X): $p_1, p_2, p_3, \ldots\ldots, p_n$

be a discrete probability distribution.

We denote the *mean* by μ and define $\mu = \dfrac{\Sigma p_i x_i}{\Sigma p_i} = \Sigma p_i x_i$ $\quad (\because \Sigma p_i = 1)$

Other names for the mean are *average or expected value E(X).*

We denote the *variance* by σ^2 and define $\sigma^2 = \Sigma p_i (x_i - \mu)^2$

If μ is not a whole number, then $\sigma^2 = \Sigma p_i x_i^2 - \mu^2$

Standard deviation $\sigma = + \sqrt{\text{Variance}}$.

The **r^th^ moment** about the mean is denoted by μ_r and defined as

$$\mu_r = \sum (x_i - \mu)^r f(x_i) \quad \text{where } f(x_i) = p_i$$

Putting, $r = 0, 1, 2, 3$ and 4, we get

$$\mu_0 = \sum f(x_i) = \sum p_i = 1$$

$$\mu_1 = \sum (x_i - \mu) f(x_i) = \sum (x_i - \mu) p_i$$

$$= \sum p_i x_i - \mu \sum p_i = \mu - \mu = 0$$

$$\mu_2 = \sum (x_i - \mu)^2 f(x_i) = \sigma^2$$

$$\mu_3 = \sum (x_i - \mu)^3 f(x_i)$$

$$\mu_4 = \sum (x_i - \mu)^4 f(x_i)$$

In practice the first four moments suffice.

Mean deviation from mean $= \sum |x_i - \mu| f(x_i)$

(b) If X is a continuous random variable with probability density function $f(x)$, then

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = E(X^2) - [E(X)]^2$$

Mean deviation from mean $= \int_{-\infty}^{\infty} | x - \mu | f(x)\, dx$

The **r^th^ moment** about the mean is defined as

$$\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r f(x) dx$$

Moments about the mean are called **central moments.**

(c) **Moment Generating Function**

Consider the function $M_a(t) = \sum p_i e^{t(x_i - a)}$ ...(1)

This function is a function of the parameter t and gives the mean of the probability distribution of $e^{t(x_i - a)}$. Expanding the exponential in equation (1), we get

$$M_a(t) = \sum p_i \left[1 + t(x_i - a) + \frac{t^2}{2!}(x_i - a)^2 + \ldots + \frac{t^r}{r!}(x_i - a)^r + \ldots\right]$$

$$= \sum p_i + t\sum p_i (x_i - a) + \frac{t^2}{2!}\sum p_i (x_i - a)^2 + \ldots + \frac{t^r}{r!}\sum p_i (x_i - a)^r + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu'_2 + \ldots + \frac{t^r}{r!}\mu'_r + \ldots \qquad \ldots(2)$$

where μ'_r is the moment of order r about a. Thus $M_a(t)$ generates moments and is, therefore, called the moment generating function (m.g.f.) of the discrete probability distribution of the variate X about the value $x = a$. We observe that μ'_r is equal to the co-efficient of $\dfrac{t^r}{r!}$ in the expansion of m.g.f. $M_a(t)$.

Alternately, μ'_r can be obtained by differentiating (2) r times with respect to t and then putting $t = 0$.

Thus, $$\mu'_r = \left[\frac{d^r}{dt^r} M_a(t)\right]_{t=0}$$

Rewriting equation (1) as

$$M_a(t) = \sum p_i e^{t(x_i - a)} = e^{-at} \sum p_i e^{tx_i} = e^{-at} M_0(t)$$

Hence the m.g.f. about the value a is e^{-at} times the m.g.f. about the origin.

If X is a continuous random variable with probability density function $f(x)$, then

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x)\, dx$$

Note 1. If X is a random variable and a, b are constants, then

$E(aX + b) = aE(X) + b$

$\text{Var.}(aX + b) = a^2\, \text{Var.}(X)$

2. Relation between central moments and moments about any arbitrary origin a

$\mu'_1 = \mu - a$

$\mu_2 = \mu'_2 - (\mu'_1)^2$

$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$

$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$

ILLUSTRATIVE EXAMPLES

Example 1. *Five defective bulbs are accidentally mixed with twenty good ones. It is not possible to just look at a bulb and tell whether or not it is defective. Find the probability distribution of the number of defective bulbs, if four bulbs are drawn at random from this lot.*

Sol. Let X denote the number of defective bulbs in 4. Clearly X can take the values 0, 1, 2, 3 or 4.

Number of defective bulbs = 5

Number of good bulbs = 20

Total number of bulbs = 25

$$P(X = 0) = P\text{ (no defective)} = P\text{ (all 4 good ones)}$$

$$= \frac{^{20}C_4}{^{25}C_4} = \frac{20 \times 19 \times 18 \times 17}{25 \times 24 \times 23 \times 22} = \frac{969}{2530}$$

$$P(X = 1) = P(\text{one defective and 3 good ones}) = \frac{^5C_1 \times {}^{20}C_3}{^{25}C_4} = \frac{1140}{2530}$$

$$P(X = 2) = P\text{ (2 defectives and 2 good ones)} = \frac{^5C_2 \times {}^{20}C_2}{^{25}C_4} = \frac{380}{2530}$$

$$P(X = 3) = P(\text{3 defectives and 1 good one}) = \frac{^5C_3 \times {}^{20}C_1}{^{25}C_4} = \frac{40}{2530}$$

$$P(X = 4) = P\text{ (all 4 defective)} = \frac{^5C_4}{^{25}C_4} = \frac{1}{2530}$$

$\therefore$ The probability distribution of the random variable X is

X :	0	1	2	3	4
P(X) :	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$.

Example 2. *A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and the variance of the number of successes.*

Sol. Let X denote the number of success. Clearly X can take the values 0, 1, 2 or 3.

Probability of success $= \frac{2}{6} = \frac{1}{3}$; Probability of failure $= 1 - \frac{1}{3} = \frac{2}{3}$

$$P(X = 0) = P\text{ (no success)} = P\text{ (all 3 failures)} = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

$$P(X = 1) = P\text{ (one success and 2 failures)} = {}^3C_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$$

$$P(X = 2) = P\text{ (two successes and one failure)} = {}^3C_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$$

$$P(X = 3) = P\text{ (all 3 successes)} = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$\therefore$ The probability distribution of the random variable X is

X :	0	1	2	3
P(X) :	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

To find the mean and variance

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	$\frac{8}{27}$	0	0
1	$\frac{12}{27}$	$\frac{12}{27}$	$\frac{12}{27}$
2	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{24}{27}$
3	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{9}{27}$
		1	$\frac{5}{3}$

Mean $\mu = \Sigma p_i x_i = 1$

Variance $\sigma^2 = \Sigma p_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}$.

Example 3. *A random variable X has the following probability function:*

Values of X, x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, *(ii) Evaluate* $P(X < 6)$, $P(X \geq 6)$, $P(3 < X \leq 6)$

(iii) Find the minimum value of x so that $P(X \leq x) > \frac{1}{2}$.

Sol. (*i*) Since $\sum_{x=0}^{7} p(x) = 1$, we have

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow \quad 10k^2 + 9k - 1 = 0 \quad \Rightarrow \quad (10k - 1)(k + 1) = 0$$

$$\Rightarrow \quad k = \frac{1}{10} \qquad [\because \; p(x) \geq 0]$$

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + ... + P(X = 5)$

$$= 0 + k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$P(X \geq 6) = P(X = 6) + P(X = 7)$

$$= 2k^2 + 7k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

$P(3 < X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6)$

$$= 3k + k^2 + 2k^2 = \frac{3}{10} + \frac{3}{100} = \frac{33}{100}$$

(iii) $P(X \le 1) = k = \frac{1}{10} < \frac{1}{2}$; $P(X \le 2) = k + 2k = \frac{3}{10} < \frac{1}{2}$

$P(X \le 3) = k + 2k + 2k = \frac{5}{10} = \frac{1}{2}$; $P(X \le 4) = k + 2k + 2k + 3k = \frac{8}{10} > \frac{1}{2}$

$\therefore$ The maximum value of x so that $P(X \le x) > \frac{1}{2}$ is 4.

Example 4. *In a lottery, m tickets are drawn at a time out of n tickets numbered from 1 to n. Find the expected value of the sum of the numbers on the tickets drawn.* (M.D.U. May 2011)

Sol. Let X denote the number on a ticket then X assumes vlaues 1, 2, 3, ..., n.

The probability of drawing a ticket out of n tickets is $\frac{1}{n}$.

$\therefore$ Values of X. x_i : 1 2 3 ...... n

$P(X = x_i)$: $\frac{1}{n}$ $\frac{1}{n}$ $\frac{1}{n}$ ...... $\frac{1}{n}$

$$\Rightarrow \quad E(x_i) = \sum p_i x_i$$

$$= 1 . \frac{1}{n} + 2 . \frac{1}{n} + 3 . \frac{1}{n} + ... + n . \frac{1}{n}$$

$$= \frac{1}{n} (1 + 2 + 3 + ... + n)$$

$$= \frac{1}{n} . \frac{n(n+1)}{2} = \frac{n+1}{2}$$

If the numbers on the m tickets drawn are $n_1, n_2, ..., n_m$ where $n_i \in \{1, 2, 3, ..., n\}$, $i = 1, 2, ..., m$, then the expected value of the sum of numbers on m tickets

$$= E(n_1 + n_2 + ... + n_m)$$
$$= E(n_1) + E(n_2) + ... + E(n_m)$$
$$= m\, E(x_i) = \frac{m(n+1)}{2}.$$

Example 5. *Is the function defined as follows a density function?*

$$f(x) = \begin{cases} e^{-x}, & x \ge 0 \\ 0, & otherwise \end{cases}$$

If so, find $P(1 \le X \le 2)$. (M.D.U. Dec. 2008)

Sol. $f(x) \ge 0$ for every x in $(-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx = 0 + \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = -(0-1) = 1$$

Since $f(x)$ satisfies the requirements for a density function, therefore, $f(x)$ is a density function.

$$P(1 \le X \le 2) = \int_1^2 e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_1^2 = e^{-1} - e^{-2}$$

$$= 0.368 - 0.135 = 0.233.$$

Example 6. *X is a continuous random variable with probability density function given by*

$$f(x) = \begin{cases} kx, & if \quad 0 \le x < 2 \\ 2k, & if \quad 2 \le x < 4 \\ -kx + 6k, & if \quad 4 \le x < 6 \end{cases}$$

Find k and mean value of X.

Sol. Since total probability = 1, we have

$$\int_0^6 f(x)\,dx = 1$$

$$\Rightarrow \quad \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx = 1$$

$$\Rightarrow \quad \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (-kx + 6k)\,dx = 1$$

$$\Rightarrow \quad k\left[\frac{x^2}{2}\right]_0^2 + 2k[x]_2^4 + \left[-k\frac{x^2}{2} + 6kx\right]_4^6 = 1$$

$$\Rightarrow \quad k(2-0) + 2k(4-2) + (-18k + 36k) - (-8k + 24k) = 1$$

$$\Rightarrow \quad 2k + 4k + 18k - 16k = 1$$

$$\Rightarrow \quad 8k = 1 \quad \therefore \quad k = \frac{1}{8}$$

$$\text{Mean value of X} = E(X) = \int_0^6 x\,f(x)\,dx$$

$$= \int_0^2 x\,.\,kx\,dx + \int_2^4 x\,.\,2k\,dx + \int_4^6 x(-kx + 6k)\,dx$$

$$= k\left[\frac{x^3}{3}\right]_0^2 + 2k\left[\frac{x^2}{2}\right]_2^4 + \left[-k\frac{x^3}{3} + 6k.\frac{x^2}{2}\right]_4^6$$

$$= k\left(\frac{8}{3}\right) + k(12) - \frac{k}{3}(152) + 3k(20)$$

$$= 24k = 24\left(\frac{1}{8}\right) = 3.$$

Example 7. *The density function of a random variable X is given by $f(x) = kx(2-x)$, $0 \le x \le 2$. Find (i) k (ii) mean (iii) variance (iv) mean deviation about the mean.*

Sol. (*i*) Since total probability = 1, we have

$$\int_0^2 f(x)\,dx = 1$$

$$\Rightarrow \quad \int_0^2 kx(2-x)\,dx = 1 \;\Rightarrow\; k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow \quad k\left(4 - \frac{8}{3}\right) = 1 \;\Rightarrow\; k\cdot\frac{4}{3} = 1 \;\therefore\; k = \frac{3}{4}$$

Hence $\qquad f(x) = \frac{3}{4}x(2-x), \quad 0 \le x \le 2$

(*ii*) $\qquad \text{Mean} = E(X) = \int_0^2 x\,f(x)\,dx = \frac{3}{4}\int_0^2 x^2(2-x)\,dx$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left(\frac{16}{3} - 4\right) = \frac{3}{4}\cdot\frac{4}{3} = 1$$

(*iii*) $\qquad \text{Var}(X) = E(X^2) - [E(X)]^2$

$$= \int_0^2 x^2 f(x)\,dx - (1)^2 = \frac{3}{4}\int_0^2 x^3(2-x)\,dx - 1$$

$$= \frac{3}{4}\left[2\cdot\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 - 1 = \frac{3}{4}\left(8 - \frac{32}{5}\right) - 1$$

$$= \frac{3}{4}\cdot\frac{8}{5} - 1 = \frac{6}{5} - 1 = \frac{1}{5}$$

(*iv*) Mean deviation about the mean

$$= \int_0^2 |x - \bar{x}|\,f(x)\,dx = \int_0^2 |x-1|\,f(x)\,dx$$

$$= \int_0^1 |x-1|\,f(x)\,dx + \int_1^2 |x-1|\,f(x)\,dx$$

$$= \int_0^1 (1-x)\cdot\frac{3}{4}x(2-x)\,dx + \int_1^2 (x-1)\cdot\frac{3}{4}x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^1 (2x - 3x^2 + x^3)\,dx + \frac{3}{4}\int_1^2 (-2x + 3x^2 - x^3)\,dx$$

$$= \frac{3}{4}\left[x^2 - x^3 + \frac{x^4}{4}\right]_0^1 + \frac{3}{4}\left[-x^2 + x^3 - \frac{x^4}{4}\right]_1^2$$

$$= \frac{3}{4}\left(\frac{1}{4}\right) + \frac{3}{4}\left(\frac{1}{4}\right) = \frac{3}{8}$$

Example 8. *Find the moment generating function of the exponential distribution $f(x) = \frac{1}{c}e^{-x/c}$, $0 \le x \le \infty$, $c > 0$. Hence find its mean and standard deviation.* (K.U.K. Dec. 2010)

Sol. The moment generating function about the origin is

$$M_0(t) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx}\cdot\frac{1}{c}e^{-x/c}\,dx$$

$$= \frac{1}{c}\int_0^\infty e^{\left(t-\frac{1}{c}\right)x}\,dx = \frac{1}{c}\left[\frac{e^{\left(t-\frac{1}{c}\right)x}}{t-\frac{1}{c}}\right]_0^\infty$$

$$= \frac{1}{c}\cdot\frac{c}{ct-1}(e^{-\infty} - 1) = \frac{1}{1-ct} = (1-ct)^{-1}$$

$$= 1 + ct + c^2t^2 + c^3t^3 + \dots$$

$$\mu'_1 = \left[\frac{d}{dt}M_0(t)\right]_{t=0} = [c + 2c^2t + 3c^3t^2 + \dots]_{t=0} = c$$

$\therefore \qquad \text{Mean} = 0 + \mu'_1 = c \qquad\qquad (\because\ \mu'_1 = \mu - a,\ \text{here},\ a = 0)$

$$\mu'_2 = \left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0} = [2c^2 + 6c^3t + \dots]_{t=0} = 2c^2$$

$\therefore$ Variance $\quad \sigma^2 = \mu_2 = \mu'_2 - (\mu'_1)^2 = 2c^2 - c^2 = c^2$

$\Rightarrow$ Standard deviation = c

EXERCISE 5.3

1. Two bad eggs are mixed accidently with 10 good ones. Find the probability distribution of the number of bad eggs in 3, drawn at random, without replacement, from this lot.
2. A die is tossed twice. Getting a number greater than 4 is considered a success. Find the variance of the probability distribution of the number of successes.
3. Two cards are drawn simultaneously from a well-shuffled deck of 52 cards. Compute the variance for the number of aces.
4. An urn contains 4 white and 3 red balls. Three balls are drawn, with replacement, from this urn. Find μ, σ^2 and σ for the number of red balls drawn.
5. Compute the variance of the probability distribution of the number of doublets in four throws of a pair of dice. (*M.D.U. May 2006*)
6. Four coins are tossed. What is the expectation of the number of heads ?
7. Suppose that X is a random variable for which E(X) = 10 and Var(X) = 25. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1.
8. A variate X has the probability distribution

x :	-3	6	9
$P(X = x)$:	1/6	1/2	1/3

Find E(X) and E(X^2). Hence evaluate $E(2X+1)^2$.

9. A random variable X has the following probability distribution :

x :	-2	-1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	k

Find the value of k and calculate mean and variance.

10. From an urn containing 3 red and 2 white balls, a man is to draw 2 balls at random without replacement, being promised ₹ 20 for each red ball he draws and ₹ 10 for each white one. Find his expectation.

11. A random variable X has the following probability distribution :

Values of X, x :	0	1	2	3	4	5	6	7	8
$p(x)$:	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(*i*) Determine the value of a. (*ii*) Find $P(X < 3)$, $P(X \geq 3)$, $P(2 \leq X < 5)$

(*iii*) What is the smallest value of x for which $P(X \leq x) > 0.5$?

12. Find the standard deviation for the following discrete distribution:

x :	8	12	16	20	24
$p(x)$:	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

13. Is the function given below a density function ?

$$f(x) = \begin{cases} 0 & \text{for} \quad x < 2 \\ \frac{1}{18}(3 + 2x) & \text{for} \quad 2 \leq x \leq 4 \\ 0 & \text{for} \quad x > 4 \end{cases}$$

Also find $P(2 \leq X \leq 3)$.

14. Find the mean and variance of the following density function :

$$f(x) = \begin{cases} x & \text{for} \quad 0 < x < 1 \\ 2 - x & \text{for} \quad 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

15. The probability density $p(x)$ of a continuous random variable is given by

$$p(x) = y_0\, e^{-|x|}, \; -\infty < x < \infty.$$

Prove that $y_0 = \frac{1}{2}$. Find the mean and variance of the distribution.

16. If $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0 & , \text{ elsewhere} \end{cases}$

represents the density of a random variable X, find E(X) and Var. (X).

17. Find the expectation and variance of the random variable X, whose p.d.f. is given by

$$f(x) = \begin{cases} 2e^{-2x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Answers

1.

X:	0	1	2
P(X):	$\frac{12}{22}$	$\frac{9}{22}$	$\frac{1}{22}$

2. $\frac{4}{9}$ 3. $\frac{400}{2873}$ 4. $\frac{9}{7}, \frac{36}{49}, \frac{6}{7}$ 5. $\frac{5}{9}$

6. 2 7. $a = \frac{1}{5}$, $b = 2$ 8. $\frac{11}{2}, \frac{93}{2}, 209$

9. $k = 0.1$, $\mu = 0.8$, $\sigma^2 = 2.232$ 10. ₹ 32

11. (*i*) $a = \frac{1}{81}$ (*ii*) $\frac{1}{9}, \frac{8}{9}, \frac{7}{27}$ (*iii*) 5 12. $2\sqrt{5}$

13. Yes, $\frac{4}{9}$ 14. $1, \frac{1}{6}$ 15. 0, 2 16. $\frac{1}{3}, \frac{2}{9}$

17. $\frac{1}{2}, \frac{1}{4}$.

5.19. THEORETICAL DISTRIBUTIONS

Frequency distributions can be classified under two heads:

(*i*) Observed Frequency Distributions.

(*ii*) Theoretical or Expected Frequency Distributions.

Observed frequency distributions are based on actual observation and experimentation. If certain hypothesis is assumed, it is sometimes possible to derive mathematically what the frequency distribution of certain universe should be. Such distributions are called **Theoretical Distributions**.

There are many types of theoretical frequency distributions but we shall consider only three which are of great importance:

(*i*) Binomial Distribution (or Bernoulli's Distribution);

(*ii*) Poisson's Distribution;

(*iii*) Normal Distribution.

BINOMIAL (OR BERNOULLI'S) DISTRIBUTION

5.20. BINOMIAL PROBABILITY DISTRIBUTION

Let there be n independent trials in an experiment. Let a random variable X denote the number of successes in these n trials. Let p be the probability of a success and q that of a failure in a single trial so that $p + q = 1$. Let the trials be independent and p be constant for every trial.

Let us find the probability of r successes in n trials.

r successes can be obtained in n trials in ${}^nC_r$ ways.

$$\therefore \quad P(X = r) = {}^nC_r P(\underbrace{SSS \ldots\ldots S}_{r \text{ times}} \; \underbrace{FFF \ldots\ldots F}_{(n-r) \text{ times}})$$

$$= {}^nC_r \underbrace{P(S)\,P(S) \ldots\ldots P(S)}_{r \text{ factors}} \; \underbrace{P(F)\,P(F) \ldots\ldots P(F)}_{(n-r) \text{ factors}}$$

$$= {}^nC_r \underbrace{p\,p\,p \ldots\ldots p}_{r \text{ factors}} \; \underbrace{q\,q\,q \ldots\ldots q}_{(n-r) \text{ factors}}$$

$$= {}^nC_r\, p^r q^{n-r}$$

Hence $\quad \mathbf{P(X = r) = {}^nC_r\, q^{n-r}\, p^r}$, where $\mathbf{p + q = 1}$ **and** $\mathbf{r = 0, 1, 2, ..., n}$.

The distribution (1) is called the *binomial probability distribution* and X is called the *binomial variate*.

Note 1. $P(X = r)$ is usually written as $P(r)$.

Note 2. The successive probabilities $P(r)$ in (1) for $r = 0, 1, 2, ..., n$ are

$$ {}^nC_0q^n,\ {}^nC_1q^{n-1}p,\ {}^nC_2q^{n-2}p^2,\ ......,\ {}^nC_np_n $$

which are the successive terms of the binomial expansion of $(q + p)^n$. That is why this distribution is called "binomial" distribution.

Note 3. n and p occurring in the binomial distribution are called the *parameters* of the distribution.

Note 4. In a binomial distribution:

(*i*) n, the number of trials is finite.

(*ii*) each trial has only two possible outcomes usually called success and failure.

(*iii*) all the trials are independent.

(*iv*) p (and hence q) is constant for all the trials.

Note 5. A binomial distribution with n trials and probability of success in each trial as p, is denoted by $B(n, p)$.

5.21. RECURRENCE OR RECURSION FORMULA FOR THE BINOMIAL DISTRIBUTION

In a binomial distribution,

$$ P(r) = {}^nC_r\, q^{n-r}\, p^r = \frac{n!}{(n-r)!\,r!}\, q^{n-r}\, p^r $$

$$ P(r+1) = {}^nC_{r+1}\, q^{n-r-1}\, p^{r+1} = \frac{n!}{(n-r-1)!\,(r+1)!}\, q^{n-r-1}\, p^{r+1} $$

$$ \therefore\quad \frac{P(r+1)}{P(r)} = \frac{(n-r)!}{(n-r-1)!} \times \frac{r!}{(r+1)!} \times \frac{p}{q} = \frac{(n-r)\times(n-r-1)!}{(n-r-1)!} \times \frac{r!}{(r+1)\times r!} \times \frac{p}{q} = \frac{n-r}{r+1}\cdot\frac{p}{q} $$

$$ \Rightarrow\quad P(r+1) = \frac{n-r}{r+1}\cdot\frac{p}{q}\, P(r) $$

which is the required recurrence formula. Applying this formula successively, we can find P(1), P(2), P(3), ..., if P(0) is known.

5.22. MEAN AND VARIANCE OF THE BINOMIAL DISTRIBUTION *(U.P.T.U. 2008)*

For the binomial distribution, $\quad P(r) = {}^nC_r\, q^{n-r}\, p^r$

$$ \text{Mean } \mu = \sum_{r=0}^{n} rP(r) = \sum_{r=0}^{n} r\,.\,{}^nC_rq^{n-r}p^r $$

$$ = 0 + 1\,.\,{}^nC_1\, q^{n-1}\, p + 2\,.\,{}^nC_2\, q^{n-2}\, p^2 + 3\,.\,{}^nC_3\, q^{n-3}\, p^3 + ... + n\,.\,{}^nC_n\, p^n $$

$$ = nq^{n-1}\, p + 2\,.\,\frac{n(n-1)}{2.1}\, q^{n-2}p^2 + 3\,.\,\frac{n(n-1)(n-2)}{3.2.1}\, q^{n-3}\, p^3 + ... + n.p^n $$

$$ = nq^{n-1}p + n(n-1)\, q^{n-2}p^2 + \frac{n(n-1)(n-2)}{2.1}\, q^{n-3}p^3 + ... + np^n $$

$$ = np\left[q^{n-1} + (n-1)q^{n-2}\,p + \frac{(n-1)(n-2)}{2.1}\, q^{n-3}\,p^2 + ... + p^{n-1}\right] $$

$$ = np[{}^{n-1}C_0q^{n-1} + {}^{n-1}C_1q^{n-2}p + {}^{n-1}C_2q^{n-3}p^2 + ... + {}^{n-1}C_{n-1}\,p^{n-1}] $$

$$ = np(q+p)^{n-1} = np \qquad (\because\ p + q = 1) $$

Hence the mean of the binomial distribution is $\boxed{\mathbf{np}}$.

$$ \text{Variance } \sigma^2 = \sum_{r=0}^{n} r^2P(r) - \mu^2 = \sum_{r=0}^{n} [r + r(r-1)]\, P(r) - \mu^2 $$

$$ = \sum_{r=0}^{n} rP(r) + \sum_{r=0}^{n} r(r-1)\, P(r) - \mu^2 = \mu + \sum_{r=2}^{n} r(r-1)\, {}^nC_rq^{n-r}p^r - \mu^2 $$

(Since the contribution due to $r = 0$ and $r = 1$ is zero)

$$ = \mu + [2\,.\,1\,.\,{}^nC_2q^{n-2}p^2 + 3.2\,.\,{}^nC_3q^{n-3}p^3 + ... + n(n-1)\,{}^nC_np^n] - \mu^2 $$

$$ = \mu + \left[2\,.\,1.\,\frac{n(n-1)}{2.1}\, q^{n-2}p^2 + 3\,.\,2.\,\frac{n(n-1)(n-2)}{3.2.1}\, q^{n-3}p^3 + ... + n(n-1)p^n\right] - \mu^2 $$

$$ = \mu + [n(n-1)q^{n-2}p^2 + n(n-1)(n-2)q^{n-3}p^3 + ... + n(n-1)p^n] - \mu^2 $$

$$ = \mu + n(n-1)\, p^2[q^{n-2} + (n-2)q^{n-3}p + ... + p^{n-2}] - \mu^2 $$

$$ = \mu + n\,(n-1)p^2\, [{}^{n-2}C_0q^{n-2} + {}^{n-2}C_1q^{n-3}p + ... + {}^{n-2}C_{n-2}p^{n-2}] - \mu^2 $$

$$ = \mu + n(n-1)p^2(q+p)^{n-2} - \mu^2 = \mu + n(n-1)p^2 - \mu^2 \qquad [\because\ q + p = 1] $$

$$ = np + n(n-1)p^2 - n^2p^2 \qquad [\because\ \mu = np] $$

$$ = np[1 + (n-1)p - np] = np[1-p] = npq. $$

Hence the variance of the binomial distribution is $\boxed{\mathbf{npq}}$.

Standard deviation of the binomial distribution is $\sqrt{npq}$.

Similarly, we can prove that

$$ \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq}; \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq} $$

$$ \text{Hence}\qquad \gamma_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}} = \frac{1-2p}{\sqrt{npq}}; \qquad \gamma_2 = \beta_2 - 3 = \frac{1-6pq}{\sqrt{npq}}. $$

Note. $\gamma_1 = \dfrac{q-p}{\sqrt{npq}} = \dfrac{1-2p}{\sqrt{npq}}$ gives a **measure of skewness** of the binomial distribution. If $p < \frac{1}{2}$, skewness is positive, if $p > \frac{1}{2}$, skewness is negative and if $p = \frac{1}{2}$, it is zero.

$\beta_2 = 3 + \dfrac{1-6pq}{npq}$ gives a **measure of the kurtosis** of the binomial distribution.

ILLUSTRATIVE EXAMPLES

Example 1. *During war, 1 ship out of 9 was sunk on an average in making a certain voyage. What was the probability that exactly 3 out of a convoy of 6 ships would arrive safely?*

Sol. p, the probability of a ship arriving safely $= 1 - \dfrac{1}{9} = \dfrac{8}{9}$; $\quad q = \dfrac{1}{9}$, $n = 6$.

Binomial distribution is $\left(\dfrac{1}{9} + \dfrac{8}{9}\right)^6$

The probability that exactly 3 ships arrive safely $= {}^6C_3\left(\dfrac{1}{9}\right)^3\left(\dfrac{8}{9}\right)^3 = \dfrac{10240}{9^6}$.

Example 2. *Assume that on the average one telephone number out of fifteen called between 2 P.M. and 3 P.M. on week-days is busy. What is the probability that if 6 randomly selected telephone numbers are called (i) not more than three, (ii) at least three of them will be busy?*

Sol. p, the probability of a telephone number being busy between 2 P.M. and 3 P.M. on week-days $= \frac{1}{15}$

$$q = 1 - \frac{1}{15} = \frac{14}{15}, \quad n = 6; \text{ Binomial distribution is } \left(\frac{14}{15} + \frac{1}{15}\right)^6$$

The probability that not more than three will be busy

$$= P(0) + P(1) + P(2) + P(3)$$

$$= {}^6C_0 \left(\frac{14}{15}\right)^6 + {}^6C_1 \left(\frac{14}{15}\right)^5 \left(\frac{1}{15}\right) + {}^6C_2 \left(\frac{14}{15}\right)^4 \left(\frac{1}{15}\right)^2 + {}^6C_3 \left(\frac{14}{15}\right)^3 \left(\frac{1}{15}\right)^3$$

$$= \frac{(14)^3}{(15)^6} [2744 + 1176 + 210 + 20] = \frac{2744 \times 4150}{(15)^6} = 0.9997$$

The probability that at least three of them will be busy

$$= P(3) + P(4) + P(5) + P(6)$$

$$= {}^6C_3 \left(\frac{14}{15}\right)^3 \left(\frac{1}{15}\right)^3 + {}^6C_4 \left(\frac{14}{15}\right)^2 \left(\frac{1}{15}\right)^4 + {}^6C_5 \left(\frac{14}{15}\right)\left(\frac{1}{15}\right)^5 + {}^6C_6 \left(\frac{1}{15}\right)^6 = 0.005.$$

Example 3. *Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?*

Sol. p = the chance of getting 5 or 6 with one die $= \frac{2}{6} = \frac{1}{3}$

$$q = 1 - \frac{1}{3} = \frac{2}{3}, \; n = 6, \; N = 729$$

since dice are in sets of 6 and there are 729 sets.

The binomial distribution is $N(q + p)^n = 729 \left(\frac{2}{3} + \frac{1}{3}\right)^6$

The expected number of times *at least three* dice showing five or six

$$= 729 \left[{}^6C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3 + {}^6C_4 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^4 + {}^6C_5 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5 + {}^6C_6 \left(\frac{1}{3}\right)^6 \right]$$

$$= \frac{729}{3^6} [160 + 60 + 12 + 1] = 233.$$

Example 4. *Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) no girl (iv) at most two girls? Assume equal probabilities for boys and girls.* (M.D.U. Dec. 2010)

Sol. Since probabilities for boys and girls are equal

$$p = \text{probability of having a boy} = \frac{1}{2}; \quad q = \text{probability of having a girl} = \frac{1}{2}$$

$$n = 4, \quad N = 800 \quad \therefore \text{ The binomial distribution is } 800\left(\frac{1}{2} + \frac{1}{2}\right)^4.$$

(i) The expected number of families having 2 *boys* and 2 girls

${}^nC_r (p)^r (q)^{n-r}$

$$= 800\, {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 800 \times 6 \times \frac{1}{16} = 300.$$

(ii) The expected number of families having *at least one boy*

$$= 800 \left[{}^4C_1 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) + {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + {}^4C_3 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4 \left(\frac{1}{2}\right)^4 \right]$$

$$= 800 \times \frac{1}{16} [4 + 6 + 4 + 1] = 750.$$

(iii) The expected number of families having no girl, *i.e.*, having 4 boys

$$= 800 \,.\, {}^4C_4 \left(\frac{1}{2}\right)^4 = 50.$$

(iv) The expected number of families having *at most two girls i.e.*, having *at least 2 boys*

$$= 800 \left[{}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + {}^4C_3 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4 \left(\frac{1}{2}\right)^4 \right] = 800 \times \frac{1}{16} [6 + 4 + 1] = 550.$$

Example 5. *Suppose X has a binomial distribution* $B\left(6, \frac{1}{2}\right)$. *Show that X = 3 is the most likely outcome.*

Sol. Here $\quad n = 6, \; p = \frac{1}{2} \quad \therefore \quad q = 1 - p = \frac{1}{2}$

$$P(0) = {}^6C_0\, q^6 = \left(\frac{1}{2}\right)^6 = \frac{1}{64}; \qquad P(1) = {}^6C_1\, q^5 p = 6\left(\frac{1}{2}\right)^6 = \frac{6}{64}$$

$$P(2) = {}^6C_2\, q^4 p^2 = 15\left(\frac{1}{2}\right)^6 = \frac{15}{64}; \qquad P(3) = {}^6C_3\, q^3 p^3 = 20\left(\frac{1}{2}\right)^6 = \frac{20}{64}$$

$$P(4) = {}^6C_4\, q^2 p^4 = 15\left(\frac{1}{2}\right)^6 = \frac{15}{64}; \qquad P(5) = {}^6C_5\, q\, p^5 = 6\left(\frac{1}{2}\right)^6 = \frac{6}{64}$$

$$P(6) = {}^6C_6\, p^6 = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

Since P(3) is the maximum among all P(r), r = 0, 1, 2, 3, 4, 5, 6, therefore, X = 3 is the most likely outcome.

Example 6. *Find the mean of the binomial distribution* $B\left(4, \frac{1}{3}\right)$.

Sol. Let X be the random variable whose probability distribution is $B\left(4, \frac{1}{3}\right)$. Then

X = 0, 1, 2, 3, 4.

x_i P_i
0 16/81
1 32/81
2 24/81
3 8/81
4 1/81

Here $n = 4,\ p = \dfrac{1}{3} \quad \therefore \quad q = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$$P(0) = {}^4C_0\left(\frac{2}{3}\right)^4 = \frac{16}{81}; \qquad P(1) = {}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) = \frac{32}{81};$$

$$P(2) = {}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{24}{81};$$

$$P(3) = {}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 = \frac{8}{81}; \quad P(4) = {}^4C_4\left(\frac{1}{3}\right)^4 = \frac{1}{81}$$

$$\text{Mean } (\mu) = \sum_{i=0}^{4} p_i x_i = \frac{16}{81}\times 0 + \frac{32}{81}\times 1 + \frac{24}{81}\times 2 + \frac{8}{81}\times 3 + \frac{1}{81}\times 4$$

$$= \frac{32 + 48 + 24 + 4}{81} = \frac{108}{81} = \frac{4}{3}.$$

Example 7. *The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99?*

Sol. Let the shooter fire n times. Here n fires are n Bernoulli trials with

$$p = \frac{3}{4} \quad \text{and} \quad q = 1 - \frac{3}{4} = \frac{1}{4}.$$

Now $\quad P(X \geq 1) > 0.99 \qquad$ (given)

$\Rightarrow \quad 1 - P(X = 0) > 0.99$

$\Rightarrow \quad P(X = 0) < 0.01$

$\Rightarrow \quad {}^nC_0\left(\frac{1}{4}\right)^n < \frac{1}{100}$

$\Rightarrow \quad 4^n > 100 \qquad$ ...(1)

The minimum value of n satisfying (1) is 4.

Hence the shooter must fire 4 times.

Example 8. *Fit a binomial distribution to the following data:*

x:	*0*	*1*	*2*	*3*	*4*
f:	*30*	*62*	*46*	*10*	*2*

(M.D.U. Dec. 2009)

Sol. The table is as follows:

x	f	fx
0	30	0
1	62	62
2	46	92
3	10	30
4	2	8
	$\Sigma f = 150$	$\Sigma fx = 192$

$$\text{Mean of observations} = \frac{\Sigma fx}{\Sigma f} = \frac{192}{150} = 1.28$$

$\Rightarrow \quad np = 1.28$

$\Rightarrow \quad 4p = 1.28 \qquad (\because n = 4)$

$\Rightarrow \quad p = 0.32 \qquad \therefore \quad q = 1 - p = 1 - 0.32 = 0.68)$

Also $\quad N = 150 \qquad [\because N = \Sigma f]$

Hence, the binomial distribution is

$$N(q + p)^n = 150\ (0.68 + 0.32)^4.$$

EXERCISE 5.4

1. Ten coins are tossed simultaneously. Find the probability of getting at least seven heads.
2. The probability of any ship of a company being destroyed on a certain voyage is 0.02. The company owns 6 ships for the voyage. What is the probability of:
   (*i*) losing one ship (*ii*) losing at most two ships (*iii*) losing none.
3. (*a*) The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of ten men now 60, at least 7 would live to be 70?
   (*b*) The probability that a man aged 50 years will die within a year is 0.01125. What is the probability that out of 12 such men, at least 11 will reach their fifty first birthday? *(M.D.U. Dec. 2011)*
4. (*a*) The incidence of occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of six workers chosen at random, four or more will suffer from the disease?
   (*b*) The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is the probability that in a group of 7, five or more will suffer from it? *(M.D.U. Dec. 2006)*
5. The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that
   (*i*) exactly two will be defective (*ii*) at least two will be defective
   (*iii*) none will be defective.
6. If the chance that one of the ten telephone lines is busy at an instant is 0.2.
   (*i*) What is the chance that 5 of the lines are busy?
   (*ii*) What is the probability that all the lines are busy?
7. If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely. *(P.T.U. 2005)*
8. A product is 0.5% defective and is packed in cartons of 100. What percentage contains not more than 3 defectives?
9. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn, one by one, with replacement, what is the probability that
   (*i*) none is white (*ii*) all are white
   (*iii*) at least one is white (*iv*) only two are white?
10. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

11. Fit a Binomial distribution for the following data and compare the theoretical frequencies with the actual ones:

x:	0	1	2	3	4	5
f:	2	14	20	34	22	8

12. If the sum of mean and variance of a binomial distribution is 4.8 for five trials, find the distribution.

13. If the mean of a binomial distribution is 3 and the variance is $\frac{3}{2}$, find the probability of obtaining at least 4 success.

14. In 800 families with 5 children each, how many families would be expected to have (*i*) 3 boys and 2 girls, (*ii*) 2 boys and 3 girls, (*iii*) no girl (*iv*) at the most two girls. (Assume probabilities for boys and girls to be equal.)

15. (*a*) In 100 sets of ten tosses of an unbiased coin, in how many cases do you expect to get

(*i*) 7 heads and 3 tails (*ii*) at least 7 heads?

(*b*) In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails?

(M.D.U. May 2011)

16. (*a*) The following data are the number of seeds germinating out of 10 on damp filter for 80 sets of seeds. Fit a Binomial distribution to this data:

x:	0	1	2	3	4	5	6	7	8	9	10	Total
f:	6	20	28	12	8	6	0	0	0	0	0	80

[**Hint.** Here $n = 10$, $N = 80$, Mean $= \frac{\Sigma fx}{\Sigma f} = 2.175$ $\therefore$ $np = 2.175$ etc.]

(*b*) Fit a Binomial distribution to the following frequency distribution:

x:	0	1	2	3	4	5	6
f:	13	25	52	58	32	16	4

(K.U.K. Dec. 2009)

17. A bag contains 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

18. A box contains 100 tickets each bearing one of the numbers from 1 to 100. If 5 tickets are drawn successively with replacement from the box, find the probability that all the tickets bear numbers divisible by 10.

19. The probability that a bomb dropped from a plane will strike the target is $\frac{1}{5}$. If six bombs are dropped, find the probability that (*i*) exactly two will strike the target, (*ii*) at least two will strike the target.

(M.D.U. May 2007)

20. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts. *(K.U.K Dec. 2010; M.D.U. Dec. 2010)*

21. In a lot of 500 solenoids 25 are defective. Find the probability of 0, 1, 2, 3, defective soleniods in a random sample of 20 solenoids. *(M.D.U. May 2008)*

Answers

1. $\frac{11}{64}$ **2.** (*i*) 0.1085, (*ii*) 0.9997, (*iii*) 0.8858

3. (*a*) 0.514 (*b*) 0.9923

4. (*a*) 0.017 (*b*) 0.0008 **5.** (*i*) 0.2301 (*ii*) 0.3412 (*iii*) 0.2833

6. (*i*) 0.02579 (*ii*) 1.024×10^{-7} **7.** 0.91854 **8.** 99.83

9. (*i*) $\frac{81}{256}$ (*ii*) $\frac{1}{256}$ (*iii*) $\frac{175}{256}$ (*iv*) $\frac{27}{128}$ **10.** $\frac{5}{2}\left(\frac{5}{6}\right)^9$

11. $100\,(.432 + 0.568)^5$ **12.** $\left(\frac{1}{5}+\frac{4}{5}\right)^5$ **13.** $\frac{11}{32}$

14. (*i*) 250, (*ii*) 250, (*iii*) 25, (*iv*) 400 **15.** (*a*) (*i*) 12 nearly (*ii*) 17 nearly (*b*) 31

16. (*a*) $80\,(0.7825 + 0.2175)^{10}$ (*b*) $200\,(0.554 + 0.446)^6$

17. $\left(\frac{9}{10}\right)^4$ **18.** 0.00001 **19.** (*i*) 0.246 (*ii*) 0.345 **20.** 323

21. 0.3585, 0.3774, 0.1887, 0.0596.

POISSON DISTRIBUTION

5.23. POISSON DISTRIBUTION AS A LIMITING CASE OF BINOMIAL DISTRIBUTION

If the parameters n and p of a binomial distribution are known, we can find the distribution. But in situations where n is very large and p is very small, application of binomial distribution is very labourious. However, if we assume that as $n \to \infty$ and $p \to 0$ such that np always remains finite, say λ, we get the Poisson approximation to the binomial distribution.

Now, for a Binomial distribution

$$P(X = r) = {}^nC_r q^{n-r} p^r$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \times (1-p)^{n-r} \times p^r$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \times \left(1-\frac{\lambda}{n}\right)^{n-r} \times \left(\frac{\lambda}{n}\right)^r \qquad \text{since } np = \lambda \quad \therefore \quad p = \frac{\lambda}{n}$$

$$= \frac{\lambda^r}{r!} \times \frac{n(n-1)(n-2)\ldots(n-r+1)}{n^r} \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\ldots\ldots\left(\frac{n-r+1}{n}\right) \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{r-1}{n}\right) \times \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}$$

As $n \to \infty$, each of the $(r-1)$ factors

$$\left(1-\frac{1}{n}\right), \left(1-\frac{2}{n}\right), \ldots\ldots, \left(1-\frac{r-1}{n}\right) \qquad \text{tends to 1. Also } \left(1-\frac{\lambda}{n}\right)^r \text{ tends to 1.}$$

Since $\underset{x \to \pm\infty}{\text{Lt}} \left(1+\frac{1}{x}\right)^x = e$, the Naperian base. $\therefore$ $\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda} \longrightarrow e^{-\lambda}$ as $n \to \infty$

Hence in the limiting case when $n \to \infty$, we have

$e = 2.718$

$$\boxed{P(X = r) = \lambda^r \frac{e^{-\lambda}}{r!} \quad (r = 0, 1, 2, 3, \ldots)} \qquad \ldots(A)$$

where λ is a finite number $= np$.

(A) represents a probability distribution which is called the Poisson probability distribution.

Note 1. λ is called the parameter of the distribution.

Note 2. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots$ to ∞.

Note 3. The sum of the probabilities $P(r)$ for $r = 0, 1, 2, 3, \ldots\ldots$ is 1, since

$$P(0) + P(1) + P(2) + P(3) + \ldots\ldots = e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \ldots\ldots$$

$$= e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\ldots\right) = e^{-\lambda} \cdot e^{\lambda} = 1.$$

5.24. RECURRENCE FORMULA FOR THE POISSON DISTRIBUTION

(U.P.T.U. 2009)

For Poisson distribution, $P(r) = \frac{\lambda^r e^{-\lambda}}{r!}$ and $P(r+1) = \frac{\lambda^{r+1} e^{-\lambda}}{(r+1)!}$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{\lambda r!}{(r+1)!} = \frac{\lambda}{r+1} \quad \text{or} \quad \boxed{P(r+1) = \frac{\lambda}{r+1} P(r)}, \; r = 0, 1, 2, 3, \ldots\ldots$$

This is called the recurrence formula for the Poisson distribution.

5.25. MEAN AND VARIANCE OF THE POISSON DISTRIBUTION

(U.P.T.U. 2006)

For the Poisson distribution, $P(r) = \frac{\lambda^r e^{-\lambda}}{r!}$

$$\text{Mean } \mu = \sum_{r=0}^{\infty} r P(r) = \sum_{r=0}^{\infty} r \cdot \frac{\lambda^r e^{-\lambda}}{r!}$$

$$= e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^r}{(r-1)!} = e^{-\lambda}\left(\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \ldots\ldots\right)$$

$$= \lambda e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\ldots\right) = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$\lambda = np$

Thus, the mean of the Poisson distribution is equal to the parameter λ.

$$\text{Variance } \sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2 = \sum_{r=0}^{\infty} r^2 \cdot \frac{\lambda^r e^{-\lambda}}{r!} - \lambda^2 = e^{-\lambda} \sum_{r=1}^{\infty} \frac{r^2 \lambda^r}{r!} - \lambda^2$$

$$= e^{-\lambda}\left[\frac{1^2 \cdot \lambda}{1!} + \frac{2^2 \cdot \lambda^2}{2!} + \frac{3^2 \lambda^3}{3!} + \frac{4^2 \lambda^4}{4!} + \ldots\ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{2\lambda}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \ldots\ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{(1+1)\lambda}{1!} + \frac{(1+2)\lambda^2}{2!} + \frac{(1+3)\lambda^3}{3!} + \ldots\ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\ldots\right) + \left(\frac{\lambda}{1!} + \frac{2\lambda^2}{2!} + \frac{3\lambda^3}{3!} + \ldots\ldots\right)\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} + \lambda\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\ldots\right)\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\,[e^{\lambda} + \lambda e^{\lambda}] - \lambda^2 = \lambda e^{-\lambda} \cdot e^{\lambda}\,(1+\lambda) - \lambda^2 = \lambda(1+\lambda) - \lambda^2 = \lambda. \; = np$$

Hence, the variance of the Poisson distribution is also λ.

Thus, the mean and the variance of the Poisson distribution are each equal to the parameter λ.

Note. The mean and the variance of the Poisson distribution can also be derived from those of the binomial distribution in the limiting case when $n \to \infty$, $p \to 0$ and $np = \lambda$.

Mean of binomial distribution is np.

$\therefore$ Mean of Poisson distribution $= \underset{n \to \infty}{\text{Lt}}\, np = \underset{n \to \infty}{\text{Lt}}\, \lambda = \lambda$

Variance of binomial distribution is $npq = np\,(1-p)$

$\therefore$ Variance of Poisson distribution $= \underset{n \to \infty}{\text{Lt}}\, np\,(1-p) = \underset{n \to \infty}{\text{Lt}}\, \lambda\left(1 - \frac{\lambda}{n}\right) = \lambda$.

ILLUSTRATIVE EXAMPLES

Example 1. *If the variance of the Poisson distribution is 2, find the probabilities for $r = 1, 2, 3, 4$ from the recurrence relation of the Poisson distribution.*

Sol. λ, the parameter of Poisson distribution = Variance = 2

Recurrence relation for the Poisson distribution is

$$P(r+1) = \frac{\lambda}{r+1} P(r) = \frac{2}{r+1} P(r) \qquad \ldots(1)$$

Now $\quad P(r) = \frac{\lambda^r e^{-\lambda}}{r!} \Rightarrow P(0) = \frac{e^{-2}}{0!} = e^{-2} = .1353$

Putting $r = 0, 1, 2, 3$ in (1), we get

$$P(1) = 2P(0) = 2 \times .1353 = .2706; \quad P(2) = \frac{2}{2} P(1) = .2706$$

$$P(3) = \frac{2}{3} P(2) = \frac{2}{3} \times .2706 = .1804; \quad P(4) = \frac{2}{4} P(3) = \frac{1}{2} \times .1804 = .0902.$$

Example 2. *Assume that the probability of an individual coalminer being killed in a mine accident during a year is $\frac{1}{2400}$. Use Poisson's distribution to calculate the probability that in a mine employing 200 miners there will be at least one fatal accident in a year.*

Sol. Here $p = \frac{1}{2400}, n = 200; \quad \therefore \lambda = np = \frac{200}{2400} = \frac{1}{12} = .083$

$$\therefore \quad P(r) = \frac{\lambda^r e^{-\lambda}}{r!} = \frac{(.083)^r e^{-.083}}{r!}$$

P(at least one fatal accident) = 1 – p (no fatal accident)

$$= 1 - P(0) = 1 - \frac{(.083)^0 e^{-0.83}}{0!} = 1 - .92 = .08.$$

Example 3. *Data was collected over a period of 10 years, showing number of deaths from horse kicks in each of the 200 army corps. The distribution of deaths was as follows:*

No. of deaths:	*0*	*1*	*2*	*3*	*4*	*Total*
Frequency:	*109*	*65*	*22*	*3*	*1*	*200*

Fit a Poisson distribution to the data and calculate the theoretical frequencies. (M.D.U. May 2009)

Sol. Mean of given distribution $= \frac{\Sigma fx}{\Sigma f} = \frac{65 + 44 + 9 + 4}{200} = \frac{122}{200} = 0.61$ m

This is the parameter (m) of the Poisson distribution.

$\therefore$ Required Poisson distribution is $N \cdot \frac{m^r e^{-m}}{r!}$ where $N = \Sigma f = 200$

$$= 200e^{-0.61} \cdot \frac{(0.61)^r}{r!} = 200 \times 0.5435 \frac{(0.61)^r}{r!} = 108.7 \times \frac{(0.61)^r}{r!}.$$

r	*P(r)*	*Theoretical frequency*
0	108.7	109
1	$108.7 \times 0.61 = 66.3$	66
2	$108.7 \times \frac{(0.61)^2}{2!} = 20.2$	20
3	$108.7 \times \frac{(0.61)^3}{3!} = 4.1$	4
4	$108.7 \times \frac{(0.61)^4}{4!} = 0.7$	1
		Total = 200

Example 4. *A car-hire firm has two cars, which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused. ($e^{-1.5} = 0.2231$)* (M.D.U. Dec. 2010)

Sol. Since the number of demands for a car is distributed as a Poisson distribution with mean $m = 1.5$.

$\therefore$ Proportion of days on which neither car is used

= Probability of there being no demand for the car

$$= \frac{m^0 e^{-m}}{0!} = e^{-1.5} = 0.2231$$

Proportion of days on which some demand is refused

= probability for the number of demands to be more than two

$$= 1 - P(x \le 2) = 1 - \left(e^{-m} + \frac{me^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right)$$

$$= 1 - e^{-1.5}\left(1 + 1.5 + \frac{(1.5)^2}{2}\right) = 1 - 0.2231\,(1 + 1.5 + 1.125)$$

$$= 1 - 0.2231 \times 3.625 = 1 - 0.8087375 = 0.1912625.$$

Example 5. *Six coins are tossed 6400 times. Using the Poisson distribution, determine the approximate probability of getting six heads x times.* (U.P.T.U. 2008)

Sol. Probability of getting one head with one coin $= \frac{1}{2}$.

$\therefore$ The probability of getting six heads with six coins $= \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$\therefore$ Average number of six heads with six coins in 6400 throws $= np = 6400 \times \frac{1}{64} = 100$

$\therefore$ The mean of the Poisson distribution = 100.

Approximate probability of getting six heads x times when the distribution is Poisson

$$= \frac{m^x e^{-m}}{x!} = \frac{(100)^x \cdot e^{-100}}{\cancel{(100)!}\; x!} \qquad x = 0, 1, 2 - - -$$

EXERCISE 5.5

1. Fit a Poisson distribution to the following:

x:	0	1	2	3	4
f:	192	100	24	3	1

2. If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2000, individuals more than two will get a bad reaction.
3. If X is a Poisson variate such that P(X = 2) = 9P(X = 4) + 90P(X = 6), find the standard deviation.
4. If a random variable has a Poisson distribution such that P(1) = P(2), find
   (*i*) mean of the distribution (*M.D.U. Dec. 2011*) (*ii*) P(4). (*P.T.U. Dec. 2005*)
5. Suppose that X has a Poisson distribution. If P(X = 2) = $\frac{2}{3}$ P(X = 1) find, (*i*) P(X = 0) (*ii*) P(X = 3).
6. A certain screw making machine produces on average 2 defective screws out of 100, and packs them in boxes of 500. Find the probability that a box contains 15 defective screws. (*M.D.U. Dec. 2005*)

$n = 500, p = \frac{2}{100}, \lambda = np = 500 \times \frac{2}{100} = 10 \qquad \frac{\lambda^{15} e^{-\lambda}}{15!} = \frac{(10)^{15} e^{-10}}{15!}$

7. A manufacturer knows that the condensers he makes contain on an average 2% defective. He packs them in boxes of 100. What is the probability that a box selected at random will contain 3 or more defective condensers? (M.D.U. May 2007)

8. Fit a Poisson distribution to the following and calculate theoretical frequencies:

x:	0	1	2	3	4
f:	122	60	15	2	1

(M.D.U. 2006, Dec. 2007, May 2008)

9. Fit a Poisson distribution to the following data given the number of yeast cells per square for 400 squares:

No. of cells per sq.:	0	1	2	3	4	5	6	7	8	9	10
No. of squares:	103	143	98	42	8	4	2	0	0	0	0

(S.V.T.U. 2007)

10. Show that in a Poisson distribution with unit mean, mean deviation about mean is $\left(\frac{2}{e}\right)$ times the standard deviation.

11. In a certain factory turning razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10000 packets. (K.U.K. Dec., 2009; Madras 2006; U.P.T.U. 2009)

12. The probability that a man aged 35 years will die before reaching the age of 40 years may be taken as 0.018. Out of a group of 400 men, now aged 35 years, what is the probability that 2 men will die within the next 5 years?

13. Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages, selected at random, will be free from errors?

14. The side effects of a certain drug cause discomfort to only a few patients. The probability that any individual will suffer from these side effects is 0.002. If the drug is given to 3000 patients, what is the probability that (*i*) exactly 3 (*ii*) 5 or more than 5 will suffer side effects? (P.T.U. Dec. 2005)

15. The probability that a man aged 50 years will die within a year is 0.01125. What is the probability that out of 12 such men, at least 11 will reach their fifty first birthday? (Given $e^{-0.135} = 0.87371$)

Answers

1. $320 \times \dfrac{e^{0.503}(9.503)^r}{r!}$ 2. 0.32 3. 1 4. (*i*) 2 (*ii*) $\dfrac{2}{3e^2}$

5. (*i*) $e^{-\frac{1}{2}}$ (*ii*) $4e^{-\frac{4}{3}}$ 6. $\dfrac{(10)^{15}e^{-10}}{(15)!} = 0.035$ 7. 0.3235

8. $121.36 \times \dfrac{(0.5)^{r!}}{r!}$, where $r = 0, 1, 2, 3, 4$

Theoretical frequencies are 121, 61, 15, 3, 0 respectively

9. Theoretical frequencies are 109, 142, 92, 40, 13, 3, 1, 0, 0, 0, 0

11. 9802, 196, 2 12. 0.01936 13. 0.4795

14. (*i*) 0.1784 (*ii*) 0.7150 15. 0.99166.

NORMAL DISTRIBUTION

5.26. NORMAL DISTRIBUTION (K.U.K. Dec. 2009; U.P.T.U. 2007)

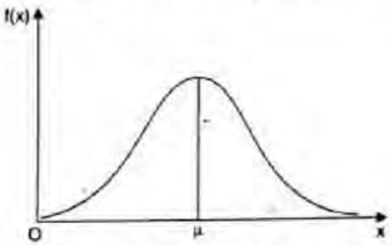
The normal distribution is a continuous distribution. It can be derived from the Binomial distribution in the limiting case when n, the number of trials is very large and p, the probability of a success, is close to $\frac{1}{2}$. The general equation of the normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where the variable x can assume all values from $-\infty$ to $+\infty$. μ and σ, called the parameters of the distribution, are respectively the mean and the standard deviation of the distribution and $-\infty < \mu < \infty$, $\sigma > 0$. x is called the normal variate and $f(x)$ is called probability density function of the normal distribution.

If a variable x has the normal distribution with mean μ and standard deviation σ, we briefly write x: $N(\mu, \sigma^2)$.

The graph of the normal distribution is called the *normal curve*. It is bell-shaped and symmetrical about the mean μ. The two tails of the curve extend to $+\infty$ and $-\infty$ towards the positive and negative directions of the x-axis respectively and gradually approach the x-axis without ever metting it. The curve is unimodal and the mode of the normal distribution coincides with its mean μ. The line $x = \mu$ divides the area under the normal curve above x-axis into two equal parts. Thus, the median of the distribution also coincides with its mean and mode. The area under the normal curve between any two given ordinates $x = x_1$ and $x = x_2$ represents the probability of values falling into the given interval. The total area under the normal curve above the x-axis is 1.



5.27. BASIC PROPERTIES OF THE NORMAL DISTRIBUTION

The probability density function of the normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

1. *The total area under normal probability curve is unity.* (M.D.U. May 2006)

Normal probability curve is given by

$$y = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Area under this curve

$$= \int_{-\infty}^{\infty} y\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$$

Put $\quad \frac{x-m}{\sigma\sqrt{2}} = t, \quad dx = \sigma\sqrt{2}\, dt$

Required area $= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sigma\sqrt{2}\, dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$

2. *The mean of the normal distribution.*

The general form of the normal curve is

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Mean $= \frac{1}{N} \Sigma fx = \frac{1}{N} \int_{-\infty}^{\infty} yx\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$

Putting $\quad \frac{x-m}{\sigma\sqrt{2}} = t, \; dx = \sigma\sqrt{2}\, dt$

$$\therefore \text{Mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (m + \sigma\sqrt{2}t)\, e^{-t^2} \sigma\sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (m + \sigma\sqrt{2}t)\, e^{-t^2}\, dt$$

$$= \frac{m}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2}\, dt$$

$$= \frac{m}{\sqrt{\pi}} \sqrt{\pi} + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} (0) = \boxed{m} \quad \because \; te^{-t^2} \text{ is an odd function of } t$$

3. *For a normal curve, the ordinate at the mean is the maximum ordinate.*

The equation of the normal curve is

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad \text{Mean} = m$$

$$\frac{dy}{dx} = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \cdot \frac{-2(x-m)}{2\sigma^2} = -\frac{N(x-m)}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$\frac{d^2y}{dx^2} = -\frac{N}{\sigma^3\sqrt{2\pi}} \left[1 \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} + (x-m) \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} \cdot \frac{-2(x-m)}{2\sigma^2}\right]$$

$$= -\frac{N}{\sigma^3\sqrt{2\pi}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} \left[1 - \frac{(x-m)^2}{\sigma^2}\right]$$

Now $\quad \frac{dy}{dx} = 0 \quad$ when $x = m$

and $\quad \left[\frac{d^2y}{dx^2}\right]_{x=m} = -\frac{N}{\sigma^3\sqrt{2\pi}} < 0.$

Hence y, the ordinate is maximum when $x = m$ i.e., the ordinate at the mean is the maximum ordinate.

4. *The mode of the normal distribution.*

The equation of the normal curve is

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Mode is the value of x corresponding to $y = y_0$, where y_0 is the maximum frequency.

Proceeding as in Example 3, y is maximum when $x = m$.

Hence **the mode = the mean = m.**

5. *The median of the normal distribution.*

If M is the median of the normal distribution, we have

$$\int_{-\infty}^{M} \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \frac{N}{2}$$

$$\Rightarrow \quad \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{M} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{m} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{m}^{M} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \tfrac{1}{2} \qquad \ldots(1)$$

Now $\quad \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{m} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\infty}^{0} e^{-t^2} (-\sigma\sqrt{2})\, dt, \quad \text{where} \; -\frac{x-m}{\sigma\sqrt{2}} = t$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2}\, dt = \frac{1}{\sqrt{\pi}} \cdot \tfrac{1}{2} \sqrt{\pi} = \tfrac{1}{2}$$

$\therefore$ From (1)

$$\frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{m}^{M} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \tfrac{1}{2}$$

$$\Rightarrow \quad \int_{m}^{M} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = 0 \quad \Rightarrow \quad M = m$$

Hence for the normal distribution, mean, median and mode coincide.

6. *The variance and standard deviation of a normal distribution.*

The equation of the normal curve is

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad \text{Mean} = m$$

Variance $= \frac{1}{N} \int_{-\infty}^{\infty} \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \cdot (x-m)^2\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-m)^2 \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$

Putting $\quad \frac{x-m}{\sigma\sqrt{2}} = t, \quad dx = \sigma\sqrt{2}\, dt$

Variance $= \dfrac{1}{\sigma\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} 2\sigma^2 t^2 . e^{-t^2} . \sigma\sqrt{2}\, dt$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2}\, dt = \frac{4a^2}{\sqrt{\pi}} \int_0^{\infty} t^2 e^{-t^2}\, dt$$

Putting $t^2 = z$, $2t\, dt = dz$

or $$dt = \frac{dz}{2\sqrt{z}}$$

$\therefore$ Variance $= \dfrac{4\sigma^2}{\sqrt{\pi}} \displaystyle\int_0^{\infty} z e^{-z} . \dfrac{dz}{2\sqrt{z}} = \dfrac{2\sigma^2}{\sqrt{\pi}} \displaystyle\int_0^{\infty} z^{1/2} e^{-z}\, dz$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} z^{3/2-1} e^{-z}\, dz = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma(\tfrac{3}{2}) = \frac{2\sigma^2}{\sqrt{\pi}} . \tfrac{1}{2}\sqrt{\pi} = \sigma^2.$$

Standard deviation $= \sqrt{\text{Variance}} = \sigma$.

7. *The points of inflexion of the normal curve.*

Let the equation of the normal curve be

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Taking logarithms

$$\log y = \log \frac{1}{\sigma\sqrt{2\pi}} - \frac{(x-m)^2}{\sigma^2}$$

Differentiating w.r.t. x

$$\frac{1}{y} . \frac{dy}{dx} = -\frac{x-m}{\sigma^2}$$

Differentiating again

$$\frac{1}{y} . \frac{d^2y}{dx^2} - \frac{1}{y^2} . \left(\frac{dy}{dx}\right)^2 = -\frac{1}{\sigma^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{y} . \left(\frac{dy}{dx}\right)^2 - \frac{y}{\sigma^2} = \frac{1}{y} . \frac{y^2(x-m)^2}{\sigma^4} - \frac{y}{\sigma^2}$$

$$= \frac{y}{\sigma^4} [(x-m)^2 - \sigma^2]$$

At a point of inflexion,

$$\frac{d^2y}{dx^2} = 0 \qquad \left(\text{and } \frac{d^3y}{dx^3} \neq 0 \text{ which can be shown}\right)$$

$\therefore$ $$(x-m)^2 = \sigma^2$$

or $$x - m = \pm\sigma \quad \text{or} \quad x = m \pm \sigma.$$

Thus, the curve has two points of inflexion, one at $m - \sigma$ and the other at $m + \sigma$, *i.e.*, at a distance from the mean, equal to the standard deviation.

8. *The mean deviation from the mean of the normal distribution is about $\frac{4}{5}$ of its standard deviation.* (M.D.U. Dec. 2011)

Let the equation of the normal curve be

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Standard deviation $= \sigma$.

Mean deviation from the mean

$$= \int_{-\infty}^{\infty} y\,|x-m|\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-m| . e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$$

$$= \frac{\sigma^2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-\frac{1}{2}z^2}\, dz, \text{ where } z = \frac{x-m}{\sigma}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} -z . e^{-\frac{1}{2}z^2}\, dz + \int_0^{\infty} z e^{-\frac{1}{2}z^2}\, dz\right] \qquad \left|\because |z| = \begin{cases} -z & \text{if } z < 0 \\ z & \text{if } z \geq 0 \end{cases}\right.$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[-\int_{\infty}^{0} t e^{-\frac{1}{2}t^2}\, dt + \int_0^{\infty} z e^{-\frac{1}{2}z^2}\, dz\right]$$

where $t = -z$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[\int_0^{\infty} z e^{-\frac{1}{2}z^2}\, dz + \int_0^{\infty} z e^{-\frac{1}{2}z^2}\, dz\right]$$

$$\left(\because \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx \text{ and } \int_a^b f(x)\, dx = \int_a^b f(z)\, dz\right)$$

$$= \sigma\sqrt{\frac{2}{\pi}} \int_0^{\infty} z e^{-\frac{1}{2}z^2}\, dz = \sigma\sqrt{\frac{2}{\pi}} \int_0^{\infty} \tfrac{1}{2} e^{-\frac{1}{2}t}\, dt, \text{ where } t = z^2$$

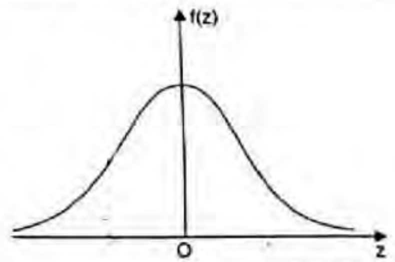
$$= \sigma\sqrt{\frac{2}{\pi}} \left[-e^{-\frac{1}{2}t}\right]_0^{\infty} = -\sigma\sqrt{\frac{2}{\pi}}\, [0 - 1]$$

$$= \sqrt{\frac{2}{\pi}} . \sigma = 0.7979\sigma = \tfrac{4}{5}\sigma \text{ (approx.)}$$

$$= \tfrac{4}{5} \times \text{standard deviation (approx.)}.$$

5.28. STANDARD FORM OF THE NORMAL DISTRIBUTION

If X is a normal random variable with mean μ and standard deviation σ, then the random variable $Z = \dfrac{X - \mu}{\sigma}$ has the normal distribution with mean 0 and standard deviation 1. The random variable Z is called the *standardized* (or *standard*) normal random variable.



$(-\infty < z < \infty)$

The probability density function for the normal distribution in standard form is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

It is free from any parameter. This helps us to compute areas under the normal probability curve by making use of standard tables.

Note 1. If $f(z)$ is the probability density function for the normal distribution, then

$$P(z_1 \le Z \le z_2) = \int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1), \quad \text{where} \quad F(z) = \int_{-\infty}^{z} f(z)\, dz = P(Z \le z)$$

The function F(z) defined above is called the *distribution function* for the normal distribution.

Note 2. The probabilities $P(z_1 \le Z \le z_2)$, $P(z_1 < Z \le z_2)$, $P(z_1 \le Z < z_2)$ and $P(z_1 < Z < z_2)$ are all regarded to be the same.

Note 3. $F(-z_1) = 1 - F(z_1)$.

ILLUSTRATIVE EXAMPLES

Example 1. *A sample of 100 dry battery cells tested to find the length of life produced the following results:*

$$\bar{x} = 12 \text{ hours}, \; \sigma = 3 \text{ hours}.$$

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life

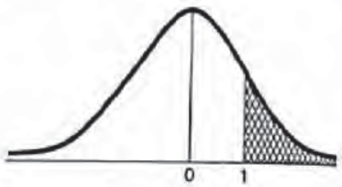
(*i*) *more than 15 hours* (*ii*) *less than 6 hours*

(*iii*) *between 10 and 14 hours?* (P.T.U. May 2005)

Sol. Here x denotes the length of life of dry battery cells.

Also $z = \frac{x - \bar{x}}{\sigma} = \frac{x - 12}{3}$.

(*i*) When $x = 15$, $z = 1$

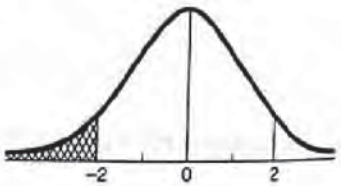


$\therefore \quad P(x > 15) = P(z > 1)$

$= P(0 < z < \infty) - P(0 < z < 1)$

$= .5 - 0.3413 = 0.1587 = 15.87\%.$

(*ii*) When $x = 6$, $z = -2$



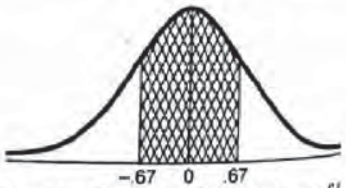
$\therefore \quad P(x < 6) = P(z < -2)$

$= P(z > 2) = P(0 < z < \infty) - P(0 < z < 2)$

$= .5 - 0.4772 = 0.0228 = 2.28\%.$

(*iii*) When $x = 10$, $z = -\frac{2}{3} = -0.67$

When $x = 14$, $z = \frac{2}{3} = 0.67$



$P(10 < x < 14)$

$= P(-0.67 < z < 0.67)$

$= 2P(0 < z < 0.67) = 2 \times 0.2487$

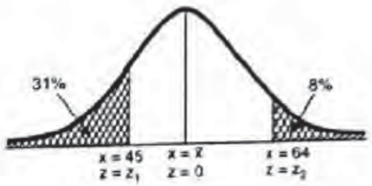
$= 0.4974 = 49.74\%.$

Example 2. *In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution.* (K.U.K., Dec. 2010)

Sol. Let $\bar{x}$ and σ be the mean and S.D. respectively.

31% of the items are under 45.

$\Rightarrow$ Area to the left of the ordinate $x = 45$ is 0.31



When $x = 45$, let $z = z_1$

$P(z_1 < z < 0) = .5 - .31 = .19$

From the tables, the value of z corresponding to this area is 0.5

$\therefore \quad z_1 = -0.5 \; [z_1 < 0]$

When $x = 64$, let $z = z_2$

$P(0 < z < z_2) = .5 - .08 = .42$

From the tables, the value of z corresponding to this area is 1.4.

$z_2 = 1.4$

Since $z = \frac{x - \bar{x}}{\sigma}$

$-0.5 = \frac{45 - \bar{x}}{\sigma}$ and $1.4 = \frac{64 - \bar{x}}{\sigma}$

$\Rightarrow \quad 45 - \bar{x} = -0.5\sigma$ ...(1)

and $\quad 64 - \bar{x} = 1.4\sigma$ ...(2)

Subtracting $\quad -19 = -1.9\sigma \quad \therefore \quad \sigma = 10$

From (1), $\quad 45 - \bar{x} = -0.5 \times 10 = -5 \quad \therefore \quad \bar{x} = 50.$

EXERCISE 5.6

1. If z is the standard normal variate, then find the following probabilities:
   (*i*) $P(1 \le z \le 2)$ (*ii*) $P(-2.3 \le z \le -1.5)$
   (*iii*) $P(-0.42 \le z \le 2)$ (*iv*) $P(z \le -0.56)$
   (*v*) $P(|z| \le 1)$ (*vi*) $P(|z| \ge 1)$
2. Let X be a random variable having a normal distribution with mean 30 and standard deviation 5. Find the probability that
   (*i*) $26 \le x \le 40$ (*ii*) $|x - 30| > 5$
   (*iii*) $|x - 24| < 8$ (*iv*) $x \ge 45$
   (*M.D.U. Dec. 2010; J.N.T.U. 2005; P.T.U. May 2006*)
3. The mean height of 500 male students in a certain college is 151 cm and the standard deviation is 15 cm. Assuming the heights are normally distributed, find how many students have heights between 120 and 155 cm?
4. An aptitude test for selecting officers in a bank is conducted on 1000 candidates. The average score is 42 and the standard deviation of score is 24. Assuming normal distribution for the scores, find
   (*i*) the number of candidates whose scores exceed 60
   (*ii*) the number of candidates whose scores lie between 30 and 60.
5. If the mean height of an Indian police inspector be 170 cm with variance 25 cm², how many inspectors out of 1000 would you expect
   (*i*) between 170 cm and 180 cm (*ii*) less than 160 cm?
6. The marks obtained by a large group of students in a final examination in statistics have a mean of 58 and a standard deviation of 8.5. Assuming that these marks are approximately normally distributed, what percentage of the students can be expected to have obtained marks from 60 to 69?

7. The income of a group of 10,000 persons was found to be normally distributed with mean = ₹ 750 p.m. and standard deviation = ₹ 50. Show that of this group about 95% had income exceeding ₹ 668 and only 5% had income exceeding ₹ 832. What was the lowest income among the richest 100? (U.P.T.U. 2005; P.T.U. Dec. 2006)
8. In a normal distribution, 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution?
9. Let X denote the number of scores on a test. If X is normally distributed with mean 100 and standard deviation 15, find the probability that X does not exceed 130.
10. It is known from the past experience that the number of telephone calls made daily in a certain community between 3 p.m. and 4 p.m. have a mean of 352 and a standard deviation of 31. What percentage of the time will there be more than 400 telephone calls made in this community between 3 p.m. and 4 p.m.?
11. Students of a class were given a mechanical aptitude test. Their marks were found to be normally distributed with mean 60 and standard deviation 5. What per cent of students scored:
    (i) more than 60 marks? (ii) less than 56 marks?
    (iii) between 45 and 65 marks?
12. In an examination taken by 500 candidates, the average and the standard deviation of marks obtained (normally distributed) are 40% and 10%. Find approximately
    (i) how many will pass, if 50% is fixed as a minimum?
    (ii) what should be the minimum if 350 candidates are to pass?
    (iii) how many have scored marks above 60%?
13. In a distribution, exactly normal, 9.85% of the items are under 40 and 89.97% are under 60. What are the mean and standard deviation of the distribution? (P.T.U. Dec. 2002)
14. The income distribution of workers in a certain factory was found to be normal with mean of Rs. 500 and standard deviation of Rs. 50. There were 228 workers getting above Rs. 600. How many workers were there in all?
15. The mean inside diameter of a sample of 500 washers produced by a machine is 5.02 mm and the standard deviation is 0.05 mm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 4.96 to 5.08 mm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.
16. Fit a normal curve to the following distribution:

x:	2	4	6	8	10
f:	1	4	6	4	1

(M.D.U. Dec. 2008, May 2011)

Answers

1. (i) 0.1359 (ii) 0.0561 (iii) 0.64
   (iv) 0.2877 (v) 0.6826 (vi) 0.3174
2. (i) 0.7653 (ii) 0.3174 (iii) 0.7231 (iv) 0.0013
3. 300 4. (i) 252 (ii) 533
5. (i) 477 (ii) 23 6. 30.67%
7. Rs. 866.50 8. $\mu = 50.3$, $\sigma = 10.33$
9. 0.9772 10. 6.06%
11. (i) 50% (ii) 21.2% (iii) 84%
12. (i) 79 (ii) 35% (iii) 11
13. $\mu = 157.29$, $\sigma = 54.05$ 14. 10,000 15. 23
16. $y = \dfrac{1}{2\sqrt{2\pi}} e^{-\frac{(x-6)^2}{8}}$